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NON-CIRCULAR BEVEL GEARS

BY

UNO OLSSON

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Preface

This investigation may be regarded as an extension of the author's book entitled "Non-circular Cylindrical Gears" (Ref. 12). Although non-circular bevel gears may not have the same practical value as non-circular cylindrical gears, it has nevertheless been the endeavour of the author to supplement his previous investigation with a similar study of non-circular bevel gears in order that a complete theory embracing all types of non-circular gear might be available.

It is the author's hope that this work may be of value not only for the designing and manufacture of non-circular bevel gears, but also, to some extent, for pure mathematics, since Chapters III—V contain a rather extensive study of spherical analytical geometry, with particular emphasis being laid on the great circle or the geodetic line (corresponding to the straight line in plane geometry), the spherical conic sections (corresponding to the second degree curves in plane geometry) and their rolling curves. Although spherical trigonometry has already been dealt with very thoroughly, it does not appear as if this is the case with spherical analytical geometry. When treating the spherical curves, the author employs spherical polar coordinates as well as a new coordinate system, entitled *great circle coordinates*. This coordinate system is analogous to rectangular coordinates in plane analytical geometry. The great circle coordinates underline in a striking manner the relationship between plane and spherical geometry.

The author wishes here to express his gratitude to ASEA for having financially supported the publication of this work. Particular thanks are due to the Directors of this company, Dr K. E. Eriksson, Dr I. Herlitz, Mr H. Liander and Mr G. Rydbeck, for having made this investigation possible.

The author is also indebted to several collaborators in, and outside, the ASEA organisation for valuable assistance with the calculations, preparation of the figures, experimental work in the workshops, typing, correction of the proofs and so forth. Special thanks are due to Mr S. Watts, of Messrs. FULLER Electric Ltd., London, who has assisted the author with the translation of the Swedish manuscript, to Mr L. Pettersson, ASEA, Ludvika, who has performed a great part of the numerical calculations, to Mr A. Olsson, the Librarian of ASEA, Ludvika, who has helped the author with the bibliography and similar matters, to Mr M. D. Blake, ASEA, Västerås, who has read the English text of the manuscript, and to Miss A. M. Bokvist, who has corrected the proofs.

Ludvika, February, 1959

Uno Olsson

Before using the book "*Non-circular Cylindrical Gears*" please make the corrections listed below.

Errata

in *Non-circular Cylindrical Gears*. Stockholm 1953. (Acta Polytechnica No. 135. Mechanical Engineering Series. Vol. 2. No. 10)

Page X, line 8: for initial *read* end
» 13, line 3 from bottom: for $= C$ *read* corresponding to C
» 16, line 8: for cot Θ *read* cot Φ
» 18, line 2 from bottom: for v *read* Θ
» 36, line 8 from bottom: for $|a|$ *read* $|a|^2$
» 40, line 10 from bottom: for ν *read* ϑ
» 47, line 8: for complex *read* imaginary
» 53, lines 5 and 6 from bottom: for $a_1 (\varkappa_L - \varkappa_R)$ and $a_1 (\varkappa_L + \varkappa_R)$ *read* $a_1 (1 - \varkappa_\nu)$ and $a_1 (1 + \varkappa_\nu)$
» 68, line 16: for $\frac{1}{i} \sin \varphi_\nu$ *read* $\frac{1}{i} \tan \varphi_\nu$
» 91, line 19: for $\cos \alpha_\nu = 1$ *read* $\cos \alpha_\nu = \nu$
» 93, line 5 from bottom: for $\tan \left(|\nu| \frac{\Delta \Theta}{2} \right)$ *read* $\tanh \left(|\nu| \frac{\Delta \Theta}{2} \right)$
» 96, line 1: for $v = |v|$ *read* $\nu = |\nu|$
» 103, line 4: for p. 192 *read* p. 212
» 107, line 6 from bottom: for n_R *read* n_L
» 108, line 13: for n_R *read* n_L
» 122, line 3: for $\frac{r}{C_{L-R}}$ *read* r
» 128, lines 12 and 14 from bottom: for arbour *read* arbor
» 130, line 13: for ψ *read* Φ
» 132, lines 1 and 14 from bottom: for arbour *read* arbor
» 184, line 8 from bottom: for arbour *read* arbor
» 189, line 10 from bottom: for Recuei *read* Recueil
» 214, $\varphi = 55.8°$: for 67.5714 *read* 67.5417

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Symbols

The numbers in brackets refer to the page where the symbol is defined or first used.

A = addendum in angular measure [126].

A as a suffix indicates that the value applies to an antipode curve [60].

a or a_1 = the abscissa at the origin [13], half the major axis of a real spherical conic section curve [16], [44], or a corresponding constant for other spherical conic sections [29], or occasionally a side in a triangle [144].

$\bar{a}$ = $90° - a$ [18].

amh = the hyperbolic amplitude (the Gudermannian) [38].

a_ν = a constant for spherical conic section rolling curves with periodic number $= \nu$, analogous to a_1 for the pure spherical conic sections [44].

$\bar{a}_\nu$ = $90° - |\varkappa_\nu||a_1|$ [77].

B_{ih} = the dedendum minus bottom clearance in angular measure [125].

b = the ordinate at the origin [13], half the minor axis of a spherical conic section curve [16], or a corresponding constant for other spherical conic sections [29], or occasionally a side in a triangle [144].

b as a suffix indicates the base curve of the tooth profile or of the cutter [132].

b_{so} = the out-of-centre setting of the cutter when machining the side cuts [130].

C = an arbitrary constant [8].

C as a suffix indicates that the value belongs to a circular bevel gear [139].

C_{L-R} and $\overline{C}_{L-R}$ = the angle between the axes of rotation for a rolling curve pair, taken positive in the direction from the curve with the suffix L to the point of contact of the curves if the radius vector of the L-curve is positive and vice versa [5], [113].

C_T = the same angle for a twin gear [8].

C_ν = the same angle if one of the curves is a single-period and the other one is a ν-period curve [5].

C_1 = a constant signifying a coordinate rotation [92].

C_2, C_4, C_6 = constants [15].

c = a constant defined by eq. (251) [115] or occasionally a side in a triangle [144].

c as a suffix for the values v_d, v_h, v_v, h and l indicates that the values are valid when cutting the tooth spaces [125].

c_b = the bottom clearance [128].

c_c = a constant obtained from Appendix 8 and used when determining the out-of-centre setting of the cutter [130].

c_{pc} = $1 + \frac{Z_c}{2}\left(1 - \cos\frac{90°}{Z_c}\right)$ = an auxiliary value used when determining h_{pc} [133].

$c_1, c_2, c_3 \ldots$ = constants [13].

$\bar{c}_1, \bar{c}_2, \bar{c}_3 \ldots$ = constants [14].

$\overline{\overline{c}}_1, \overline{\overline{c}}_2, \overline{\overline{c}}_3 \ldots$ = constants [14].

$\overline{\overline{\overline{c}}}_1, \overline{\overline{\overline{c}}}_2, \overline{\overline{\overline{c}}}_3 \ldots$ = constants [15].

D_c = the diameter of the setting disc used when cutting the tooth spaces [128].

D_t = the diameter of the setting disc used when cutting the tips of the teeth [146].

d = the whole depth of the tooth space for which the cutter is designed [128].

E. = the east pole in great-circle coordinates [10].

$E(\alpha)$ = the complete elliptic integral of the second kind [27].

$E(\alpha, \varphi)$ = the incomplete elliptic integral of the second kind [27].

e = the eccentricity of a spherical conic section (when not used in a power expression) [16], the base of the natural system of logarithms (when used in a power expression) [38].

e as a suffix indicates an end value [43] or a value belonging to a spherical ellipse [26].

$\bar{e}$ = $\dfrac{90° - ea}{90° - a}$ [18].

F = the face width of the tooth [131].

$F(\alpha)$ = the complete elliptic integral of the first kind [27].

$F(\alpha, \varphi)$ = the incomplete elliptic integral of the first kind [27].

f = the distance from the centre of the dividing head to the apex of the pitch cone [128].

f_1 = the distance from the centre of the dividing head to the locating face for the boss of the wheel [129].

f_2 = the distance from the outer side of the wheel boss to the apex of the pitch cone [129].

g_p = the chord thickness of cutter used, measured at pitch line [130].

g_{pc} = the chord thickness at pitch line for a normal cutter [132].

h = the machine setting (height setting) when milling [127].

$\bar{h}$ = the value of h if measured from the position where the tip of the cutter coincides with the centre of the dividing head [128].

h_{pc} = the calliper setting of height when measuring chord thickness of cutter at pitch line [133].

i = the unit of imaginaries [17].

k_i = a constant used when determining a single-turn mating curve to a given rolling curve [112].

k_p = half the chord of the tooth space at pitch line [134].

k_{pc} = half the chord at the pitch line for a normal cutter [132].

k_{qc} = half the chord of a normal cutter at the radius ϱ_{qc} [132].

k_1 and k_2 = constants used in the equations of spherical conic section rolling curves [47].

$\bar{k}_1$ and $\bar{k}_2$ = adjusted values of k_1 and k_2 [115].

k_3 = an auxiliary angle, used when determining ν [Appendix 2].

k_4 = a constant used when determining ϑ_s for a spherical conic section rolling curve [Appendix 2].

L as a suffix indicates one wheel, generally the left-hand one [3] or the left-hand cut [139].

l = the machine setting (cross-setting) when milling [127].

ln = the natural logarithm [40].

M as a suffix indicates that the value applies to a main curve [60].

m = the module [119], order number [111] or an integer [3].

max as a suffix indicates a maximum value [25].

min as a suffix indicates a minimum value [25].

mm = millimetre [147].

N. = the north pole in great circle coordinates [10].

n = the number of periods per turn (periodic number) [3], the tooth-space number [122].

O = the point of origin [10].

p = a positive parameter in the elliptic integral of the third kind [27].

p as a suffix indicates that the value refers to the pitch curve [120] or to the large end of the tooth or tooth space [137].

p_1 = a negative parameter in the elliptic integral of the third kind [26].

q as a suffix indicates that the value refers to the small end of the tooth or tooth space [137].

R = the absolute value of the spherical radius vector [73], [108].

R as a suffix indicates one wheel, generally the right-hand one [3] or the right-hand cut [139].

R_{sph} = the radius of the sphere on which the pitch curve lies = the length of the generatrix of the pitch cone [119].

r and $\bar{r}$ = the spherical radius vector [3], [113].

r_c = the spherical radius vector to the theoretical bottom of the tooth space [125].

r_l and r_ν = the focal radii [19].

r_t = the spherical radius vector to the tip of the tooth [146].

r_ν = the spherical radius vector for a ν-period rolling curve [44].

S. = the south pole in great circle coordinates [10].

S = the arc length in angular measure [26] in connection with which the following suffixes are used:

- e for the spherical ellipse [26] or for an end value [43].
- ei for the imaginary spherical ellipse [32].
- exp for the spherical exponential curve [41].
- hi for the imaginary spherical hyperbola [36].
- n for tooth-space number n [123].
- pc for the whole perimeter of the pitch curve [119].
- s for an initial or starting value [123].
- $^1/_4$ for a quadrant of a spherical conic section from $\varphi = 0$ to $\varphi = 90°$ [28].

s = the arc length (angular measure) of a spherical rolling curve from an arbitrarily chosen initial position to the point of contact of the rolling curves, taken in such a direction that $\frac{ds}{d\Theta}$ has the same sign as r [5].

s as a suffix indicates an initial or starting value [43].

T as a suffix indicates that the value belongs to a spherical twin rolling curve or a twin bevel gear [8].

t as a suffix to the values v_d, v_h, v_v, h and l indicates that the values are valid when cutting the contour of the wheel (the tips of the teeth) [146].

$U_{L/R}$ and $U_{R/L}$ = the mean gear ratio when the L, or R, wheel is driving [3].

u = an auxiliary value related to the hyperbolic argument [30] or the instantaneous ratio [110].

$u_{L/R}$ and $u_{R/L}$ = the instantaneous ratio when the L, or R, wheel is driving [3].

$\bar{u}_{L/R}$ = an adjusted value of $u_{L/R}$ [115].

u_T = the instantaneous ratio for a twin bevel gear [8].

V = the cutting angle when cutting cylindrical bevel gear wheels [143].

v = an auxiliary angle [24].

v_d = the indexing of the dividing head when milling [124].

v_h = the setting of the rotary milling table when milling [124].

v_v = the tilting of the arbor of the dividing head when milling [124].

v_x, v_y = auxiliary values used when determining the setting of the circular feed device [134].

v_z = an auxiliary angle used when determining Δv_{dc} [139].

W. = the west pole in great circle coordinates [10].

w = an auxiliary angle approaching infinity [38].

X = the distance (angle) from a point (x, y) in great circle coordinates to the y-axis [10].

x, $\bar{x}$, $\bar{\bar{x}}$, $\bar{\bar{\bar{x}}}$, = great circle coordinates [10], [11], [14], [15].

Y = the distance (angle) from a point (x, y) in great circle coordinates to the x-axis [10].

$y, \bar{y}, \bar{\bar{y}}, \bar{\bar{\bar{y}}},$ = great circle coordinates [10], [11], [14], [15].

Z = the number of teeth [119].

Z_c = the number of teeth for which a cutter is designed when milling circular gears [120].

Z_{osc} = the number of teeth for a circular bevel wheel where half the apex angle of the pitch cone is $= \varrho_p$ [134].

z = an auxiliary angle used when determining a single-turn mating curve to a given curve [112] or occasionally a substitution for tan R [108].

α and α_1 = the eccentricity angle for a spherical conic section curve [22], [44], modular angle in the elliptic integrals [27] or occasionally an angle in a triangle [144].

$\bar{\alpha}$ or $\bar{\alpha}_\nu$ = an auxiliary angle replacing α or α_ν in those cases where α or α_ν is imaginary [34], [77].

α_{L+R} = the resultant angle of eccentricity for a spherical conic section rolling curve pair [102].

α_p = half the apex angle of the pitch cone for a circular bevel gear [139].

α_ν = the rotation of the great circle coordinate system about the origin [12].

α_T = an angle defined by $\tan\frac{90^\circ + \alpha_T}{2} = \tan^3\frac{90^\circ + \alpha_\nu}{2}$ = the angle α_{L+R} for a twin rolling curve pair [46].

$\bar{\alpha}_T$ = $180^\circ - \alpha_T$ [77].

α_x = the rotation of the great circle coordinate system about the N. and S. poles [11].

α_y = the rotation of the great circle coordinate system about the E. and W. poles [12].

α_ν = the eccentricity angle for a spherical conic section rolling curve with periodic number $= \nu$ [43].

α' = $90^\circ - \alpha$ [32].

β = the parametric angle in Heuman's lambda-integral [27], or occasionally an angle in a triangle [144].

β_s = the angle between the radii of the cutter to the bottom of the involute profile and to the end of the chord k_{qc} [133].

γ = the angle of displacement of a rolling curve measured clockwise [5], or occasionally an angle in a triangle [144].

ΔB_{th} = a change in the value of B_{th} when determining the settings of the milling machine [138].

Δh_c = a change in the value of h_c when determining the settings of the milling machine [138].

ΔS = the tooth pitch along the pitch line in radian measure [122].

ΔS_ν = the arc length in radian measure from the last tooth space on an arc to the end of the arc [123].

ΔV = a smaller correction when calculating the cutting angle V for cylindrical bevel gear wheels [142].

Δv_{dC} = the change in the setting v_{dC}, valid for the central cut, when milling the side cuts for cylindrical bevel gear wheels [140].

Δv_{hc} = the change in the setting v_{hc}, valid for the central cut, when milling the side cuts of the tooth spaces [130].

$\Delta\Theta$ = $\Theta_\varepsilon - \Theta_s$ [108].

$\Delta\vartheta$ = the difference between ϑ for two adjacent tooth spaces [145].

$\Delta\varphi$ = $\sqrt{1 - \sin^2\alpha\,\sin^2\varphi}$ [26].

$\Delta'\varphi$ = $\sqrt{1 - \sin^2\alpha' \sin^2\varphi}$ [32].

δ_p = an auxiliary value used when determining the settings of the milling machine [137].

δ_q = an auxiliary value used when determining the settings of the milling machine [137].

Θ = the polar angle [3].

Θ_c = the polar angle for r_c [125].

Θ_t = the polar angle for r_t [146].

ϑ, $\bar{\vartheta}$ = the polar angle [21], [38].

$\varkappa_\nu$ = a dimensionless constant [44].

$\Lambda(\alpha, \beta)$ = Heuman's complete lambda-integral [28].

$\Lambda(\alpha, \beta, \varphi)$ = Heuman's incomplete lambda-integral [27].

$\Lambda_0(\alpha, \beta)$ = $\frac{2}{\pi}\Lambda(\alpha, \beta)$ [28].

ν = the number of periods per turn (periodic number) for a spherical conic section rolling curve [43].

ν or corresponding number as a suffix indicates that the value belongs to a ν-period curve [43].

$\Pi(p, \alpha)$ = the complete elliptic integral of the third kind [27].

$\Pi(p, \alpha, \varphi)$ = the incomplete elliptic integral of the third kind [27].

ϱ = the radius of curvature [120], [144].

ϱ_{bc} = the radius of the base circle of the cutter used [132].

ϱ_{osc} = the radius of the osculating circle to the pitch line [134].

ϱ_p = the spherical radius of curvature of the pitch curve [120].

ϱ_{pc} = the radius of the pitch circle of the cutter used [131].

ϱ_{qc} = the radius of the cutter used, corresponding to the pitch curve at the small end of the teeth [131].

σ_{bc} = half the angle of the tooth space at the base circle for a full-width cutter [133].

Φ = the angle from the radius vector to the normal to the curve [24].

$\bar{\Phi}$ = an angle defined by $\tan\bar{\Phi} = \frac{\tan\Phi}{\cos\tau}$ [108].

Φ_c = the angle between the radius r_c and the normal to the pitch curve through the end of r_c [125].

Φ_t = the angle between the radius r_t and the normal to the pitch curve through the end of r_t [146].

φ = the parametric angle in the equations of the spherical conic section rolling curves [21], the amplitude angle in the elliptic integrals [27].

φ' = $90° - \varphi$ [37].

ψ = the pressure angle [132].

0, 1, 2, 3, ... as a suffix to ϑ, r, a, α, φ indicates that the magnitude belongs to a 0, 1, 2, 3, ... period spherical conic section rolling curve [43].

CHAPTER I

INTRODUCTION

In the book "Non-circular Cylindrical Gears", hereafter referred to as NCCG, published as Vol. 2, No. 10 in Acta Polytechnica, Mechanical Engineering Series (Ref. 12)[1], the author investigated the design and manufacture of non-circular cylindrical gears. In this edition of "Non-circular Bevel Gears" the author has investigated in a similar manner the design and manufacture of non-circular bevel gears.

In previous publications the author has found only one concrete example of a non-circular bevel gear, namely in REULEAUX: "Lehrbuch der Kinematik", Vol. 2 (Ref. 4), where a non-circular bevel gear having a mean gear ratio of $^1/_2$ is illustrated, and which it is said was intended to be used for a bicycle. The theory of rolling cones, i.e. cones which can roll on their surfaces without sliding and with a constant angle between their axes of rotation, has also been investigated to some extent by SCHÖNEMANN (Ref. 2), DISTELI (Ref. 3) and PESQUEIRA (Ref. 9).

In the following investigations of non-circular bevel gears and their rolling curves the concept of rolling cones is in general not used, but instead the concept of spherical rolling curves has been applied, i.e. curves in a spherical surface which can roll on each other without slipping, and with a constant angle between their centres of rotation. The corresponding rolling cones have these spherical curves as base, and their apex in the centre of the sphere. In the derivation of the equations of the spherical curves and their rolling curves, spherical geometry has been used and not solid geometry. The spherical surface in which the curves lie is therefore the only geometrical dimension considered. An arbitrary curve in a plane may be replaced by approximation, with any desired degree of accuracy, by a series of arcs of conic section rolling curves, and in a similar manner an arbitrary spherical curve can be replaced by a series of arcs of spherical conic section rolling curves, as will be shown in the following. A spherical conic section curve has the same properties as plane conic section curves, namely the sum of, or the difference between, the radius vectors from an arbitrary point on the curve to two fixed points (the foci) is constant.

After giving some definitions and some general laws for spherical rolling curves, the spherical conic sections and their rolling curves are dealt with. Similarly as in the case of the plane conic section rolling curves, it will be shown also that the rolling curves to the spherical conic section rolling curves are themselves spherical conic section rolling curves. The two spherical rolling curves or pitch curves in a bevel gear can therefore both be built up from arcs of spherical conic section rolling curves, and it will be shown how the equations of the rolling curves may be derived for an arbitrarily chosen gear ratio curve. Finally, the manufacture of non-circular bevel gears will also be treated.

[1] See bibliography on page 191.

CHAPTER II

GENERAL LAWS FOR, AND THE CONSTRUCTION OF, SPHERICAL ROLLING CURVES, i. e. PITCH CURVES FOR NON-CIRCULAR BEVEL GEARS

A. Some definitions

By *spherical rolling curve pair, wheel pair* or *gear* in what follows is meant any pair of spherical curves or bevel gears which can roll on each other without slipping and with constant angle ($\neq 0$) between two points on the spherical surface having fixed positions, each in relation to one of the two curves. Each of the curves forming a spherical rolling curve pair is called a *spherical rolling curve*. One curve in a rolling curve pair is called a *mating curve* in relation to the other. The two points in fixed relation to the curves are called *centres of rotation* in the case of rolling curves and *shaft centres* or simply *shafts* in the case of bevel gears. By the *outside* of the rolling curve is meant the side turned away from the centre of rotation and by its *inside* that side which is turned towards the centre of rotation. In general, the same designations and definitions are used for spherical rolling curves as those used in NCCG (Ref. 12) for plane rolling curves, elucidated if necessary by the attribute spherical or conical in order to avoid misunderstanding. Regarding the distance between the centres of rotation, however, the definitions differ to some extent. The distance between the centres of rotation of the rolling curves is given, in the case of plane rolling curves, as a length. In spherical geometry, however, all dimensions are given as angles. Thus the distance between the centres of rotation is given as the angle between the radii of the sphere to the two centres of rotation on the spherical surface. This dimension is called the *angle between the centres of rotation*. The absolute value of the angle between the centres of rotation is always taken as $< 180°$. For spherical rolling curves we have always two points on the spherical surface which both can be regarded as the centre of rotation, one being the antipode of the other. Sometimes it is necessary to define which of these two points is considered. If the curve is given in analytic form, the origin is considered as the centre of rotation.

A spherical rolling curve pair may at times be so shaped that the rolling takes place on the outside of both curves. The rolling curve pair and the rolling curves are then said to have *external meshing*, and the corresponding bevel gear is said to form an *external gear*. The rolling curve pair may also be such that the outside of one curve rolls on the inside of the other, in which case the rolling curve pair is said to have *internal meshing*. One of the rolling curves has then external meshing and the other one internal meshing. Corresponding bevel gears do not exist. Finally, a rolling curve pair may be such that the inside of one

rolling curve rolls on the inside of the other. The rolling curve pair is then said to have *double internal meshing*. We do not either have any corresponding bevel gears for this rolling case. An external meshing changes to an internal meshing and vice versa if the centre of rotation for the curve having the greater value of the radius is replaced by its antipode, i.e. the absolute value of the angle between the centres of rotation is replaced by the supplementary angle.

In what follows the equations for the rolling curves will, as a rule, be written in polar coordinates in the spherical surface and with the centre of rotation as pole, thus in the form

$$r = r(\Theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where $r = r(\Theta)$ = the geodetic radius or the radius vector (an arc of a great circle) measured as the angle between the radii of the sphere to the origin and to the point in question on the curve

Θ = polar angle

If $r(\Theta + m\ 360°) = r(\Theta)$, where m is an integer, the curve is said to be a *single-turn curve;* otherwise it is said to be a *multi-turn curve*. A rolling curve is said to be *single-period* if the same value of r recurs at intervals of 360° of the polar angle and *n-period* if the same value for r recurs at intervals of $\frac{360°}{n}$, where n is the number of periods per turn, in the following referred to as the *periodic number*.

Magnitudes applying to one of the rolling curves, generally the left-hand one, are indicated as a rule by the suffix L, and the curve is named the *L-curve*. Magnitudes applying to the other curve are indicated by the suffix R and this curve is called the *R-curve*. Thus the periodic number for the L-curve is denoted by n_L and for the R-curve by n_R.

If the two curves in a rolling curve pair (angle between the centres of rotation $\neq 0$) are identical, and if the absolute value of the mean gear ratio equals unity, the rolling curve pair is called a *twin rolling curve pair* and the corresponding gear a *twin gear*. The curves or wheels forming the twin rolling curve pair are called *twin rolling curves* and the corresponding wheels *twin wheels*. If the two rolling curves in a rolling curve pair are not identical, but each of the rolling curves is a twin rolling curve, the rolling curve pair is said to be of *twin type*. If one rolling curve in a rolling curve pair is the mirror image of the other and the absolute value of the mean gear ratio = 1, the rolling curve pair is said to be a *mirror rolling curve pair*. This type of rolling curve pair has not as great significance for bevel gears as for circular gears and will not be treated further.

By *instantaneous gear ratio* is meant the ratio between the angular velocities (with sign) of the driving rolling curve or wheel and the driven rolling curve or wheel. By *mean gear ratio* is meant the ratio between the periodic numbers of the driven and the driving rolling curves or wheels, taken with positive sign for internal meshing and negative sign for external meshing.

The following symbols are used for the gear ratios:

$u_{L/R}$ = the instantaneous gear ratio with the L-wheel assumed to be the driving wheel

$u_{R/L}$ = the instantaneous gear ratio with the R-wheel assumed to be the driving wheel

$U_{L/R}$ = the mean gear ratio with the L-wheel assumed to be the driving wheel

$U_{R/L}$ = the mean gear ratio with the R-wheel assumed to be the driving wheel

All ratios are given a sign, positive for internal meshing and negative for external meshing.

The following relations, which are valid for cylindrical gears, obviously also apply for spherical rolling curve pairs and bevel gears

$$u_{L/R} = \frac{1}{u_{R/L}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$U_{L/R} = \frac{1}{U_{R/L}} = \pm \frac{n_R}{n_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

the positive sign applying to internal meshing and the negative sign to external meshing. For bevel gears, however, which only have external meshing the negative sign alone needs to be considered.

A curve which gives $u_{L/R}\left(=\frac{1}{u_{R/L}}\right)$ as a function of Θ_L is called the *ratio curve with respect to the L-wheel*. A curve which gives $\frac{1}{u_{L/R}}\ (= u_{R/L})$ as a function of Θ_L is called the *reciprocal ratio curve with respect to the L-wheel*. Similarly the *ratio curve with respect to the R-wheel* is a curve which shows $u_{R/L}\left(=\frac{1}{u_{L/R}}\right)$ as a function of Θ_R, and the *reciprocal ratio curve with respect to the R-wheel* is a curve which shows $\frac{1}{u_{R/L}}\ (= u_{L/R})$ as a function of Θ_R.

It is always possible to construct for a given spherical rolling curve of any shape a mating curve with an arbitrary angle between the centres of rotation. The family of all the infinite number of mating curves with different angles between the centres of rotation is called the *rolling curve family* of the given curve, which latter may be called the *basic curve*. If the mating curve forms internal meshing with a given basic curve, it is called an *internal rolling curve*, and if it forms external meshing with a basic curve, it is called an *external rolling curve*. It may also happen that the mating curve at certain angular positions rolls with internal meshing, and at other angular positions with external meshing on the basic curve. The mating curve is then called a *mixed rolling curve* or a *rolling curve with imaginary periodic number*, the latter because in the case of spherical conic section rolling curves the periodic number of the curve appears in imaginary form in the equation of the curve.

If the basic curve is one of the generally known curves, its mating curves may suitably be called by the name of the basic curve. Thus, if the basic curve is a spherical conic section curve, the mating curves will be called *spherical conic section rolling curves*.

B. Conditions required for two spherical curves to form a spherical rolling curve pair

a) General conditions

For spherical rolling curve pairs the same fundamental law as for plane rolling curves is valid, namely: *the point of contact of the rolling curves falls on the line joining the centres of rotation*. With the aid of this fundamental law, we obtain the following condition for two spherical rolling curves to form a rolling curve pair, which law is the same as for plane rolling

curves: With external meshing, the arithmetic sum of, and with internal meshing the difference between, the radius vectors to the point of contact of the rolling curves shall be constant and equal to the angle between the centres of rotation.

The following symbols are introduced:

C_{L-R} = angle between the centres of rotation taken positive in the direction from the centre of rotation of the L-curve to the point of contact of the rolling curves, if the radius vector of the L-curve is positive and vice versa

$|C_{L-R}|$ = the absolute value of the angle between the centres of rotation

C_ν and $|C_\nu|$ = angle between the centres of rotation as above, if one rolling curve is a single-period and the other rolling curve a ν-period curve

s = length of arc measured in radians from an arbitrarily chosen initial point to the momentary point of contact of the rolling curves. Thus s is the periphery rolled over and is expressed with a sign in such a way that $\frac{ds}{d\Theta}$ will always have the same sign as r for all curves in a rolling curve family. Thus, if r changes sign, the positive direction for s is also reversed.

r_L = radius vector (angular measure) for the L-curve

$r_L(s)$ = the same radius vector as a function of s

r_R = radius vector (angular measure) for the R-curve

$r_R(s)$ = the same radius vector as a function of s

γ_L = the angle of displacement of the L-curve, taken clockwise

γ_R = the angle of displacement of the R-curve, taken clockwise

If the sign for r_R is selected in a definite manner, namely so that r_R has the same sign as r_L with internal meshing, but opposite sign to r_L with external meshing and double internal meshing, then the condition required for two curves to form a rolling curve pair, irrespective of the kind of meshing, can always be written in the form (see Fig. 1).

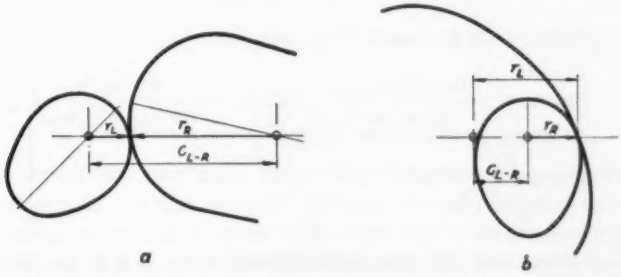


Fig. 1. The relation between the radii r_L and r_R and the angles between the axes of rotation C_{L-R}, namely $C_{L-R} = r_L - r_R$, r_R in Fig. 1 a is negative.

$$r_L(s) - r_R(s) = C_{L-R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

or, after differentiating,

$$r'_L(s) = r'_R(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

These conditions are in form the same as for plane rolling curves.

Thus, if the one rolling curve, e.g. the L-curve, is given, eq. (4) or (5) will define the mating curve as a function of s. If a rolling curve is rotating in the positive direction (counter-clockwise) the point of contact moves in the negative direction and vice versa. If the polar axes fixed to the wheels concerned pass through the point of contact of the wheels in the initial position, where $\gamma_L = \gamma_R = 0$, and have the same positive direction as the angle between the centres of rotation, and if the sign for r_R is selected as stated above, then the angles of displacement γ_L and γ_R equal the polar angles Θ_L and Θ_R. As can be understood from Fig. 1, the condition (5) implies that we must have

$$d\gamma_L \sin r_L = d\gamma_R \sin r_R$$

or

$$\frac{d\gamma_L}{d\gamma_R} = \frac{d\Theta_L}{d\Theta_R} = u_{L/R} = \frac{\sin r_R}{\sin r_L} = \frac{\sin (r_L - C_{L-R})}{\sin r_L} = \frac{\sin r_R}{\sin (r_R + C_{L-R})} \quad \ldots\ldots \quad (6)$$

Together with eq. (4) it is now possible to write the condition for a spherical rolling curve pair in the form below

$$\left.\begin{aligned} & r_L - r_R = C_{L-R} \\ & \frac{d\gamma_L}{d\gamma_R} = \frac{\sin r_R}{\sin r_L} = \frac{\sin (r_L - C_{L-R})}{\sin r_L} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

This obvious condition simply means that the angle between the centres of rotation must equal the difference between the radii to the point of contact and that the gear ratio must equal the ratio between the sines for the radii. Eq. (7) may be rewritten as

$$\left.\begin{aligned} & r_L - r_R = C_{L-R} \\ & \gamma_R = \int_0^{\gamma_L} \frac{\sin r_L}{\sin (r_L - C_{L-R})} d\gamma_L \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Eliminating r_L from eq. (8) by means of eq. (6) gives

$$\left.\begin{aligned} & \tan r_R = \frac{u_{L/R} \sin C_{L-R}}{1 - u_{L/R} \cos C_{L-R}}, \quad \tan r_L = \frac{\sin C_{L-R}}{\cos C_{L-R} - u_{L/R}} \\ & \gamma_R = \int_0^{\gamma_L} \frac{1}{u_{L/R}} d\gamma_L \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9)$$

from which the rolling curve pair may be determined if C_{L-R} is given, and the gear ratio curve is known in the form $u_{L/R} = u_{L/R}(\gamma_L)$, i.e as a function of γ_L.

As mentioned above, γ_L and γ_R equal the polar angles Θ_L and Θ_R of the curves, provided that the polar axes are suitably located and the sign for r_R is selected in a suitable manner. Often, however, the equation of the R-curve thus derived does not appear in a suitable form, but it can be made so by rotating the coordinate axes and changing the sign of r_R if r_R does not appear with suitable sign. This change can be permitted because a curve in polar coordinates will remain unchanged if the sign of the radius vector is changed, although the polar axis then will be rotated 180°.

For special rolling curves, such as single-turn curves, twin rolling curves and mirror rolling curves, further conditions have to be considered.

In the same way as for plane rolling curves, it can be shown that all spherical rolling curves belonging to the same rolling curve family and which are of the same type, either all being external rolling curves or all being internal rolling curves, can form rolling curve pairs with internal meshing. Further, all internal rolling curves can form rolling curve pairs with external meshing with all external rolling curves.

b) Conditions for single-turn spherical rolling curves

The condition required for a spherical rolling curve to be a single-turn rolling curve, either single-period or multi-period, is contained in the definition, namely that we must have

$$r(\Theta + m\ 360°) = r(\Theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where m is an integer.

The above condition means that the curve must be periodic with the periodic number equal to an integer.

If the L-curve is single-turn, we then obtain from eqs. (3) and (8) or (9) the condition required for the R-curve also to be single-turn

$$\frac{360°}{U_{L/R}} = \int_0^{360°} \frac{1}{u_{L/R}}\, d\Theta_L = \int_0^{360°} \frac{\sin r_L}{\sin (r_L - C_{L-R})}\, d\Theta_L \quad \ldots\ldots\ldots\ldots \quad (11)$$

where the mean gear ratio $U_{L/R} = \pm \frac{n_R}{n_L}$, in which the periodic numbers n_R and n_L must both be integers.

c) Conditions for spherical twin rolling curves

Twin rolling curves and twin gears are of importance for the practical designing of gears, since both wheels can then be made identical. The derivation of the condition for spherical twin rolling curves can be made in the same way as for plane twin rolling curves in NCCG (Ref. 12) on page 12 and following. Therefore, the derivations are not repeated here but only the results are given.

For plane rolling curve pairs we can have the mean gear ratio equal both to $U = -1$, e.g. two ellipses, and to $U = +1$, e.g. two hyperbolas. In the latter case, however, r becomes infinite for some parts of the curves. This, however, cannot occur for spherical curves. Therefore, spherical twin rolling curve pairs can always be considered to have a mean gear ratio $U = -1$.

One of the following four conditions must be fulfilled for such a spherical twin rolling curve.

Condition 1

$$r(C-s)+r(s)=C_{L-R}=C_T \quad \ldots\ldots (12)$$

or

$$r'(C-s)=r'(s) \quad \ldots\ldots (13)$$

where C is a constant which may be chosen arbitrarily, and the arc length s may be measured from an arbitrary initial position.

Condition 2

$$r(s)+r(-s)=C_{L-R}=C_T \quad \ldots\ldots (14)$$

or

$$r'(-s)=r'(s) \quad \ldots\ldots (15)$$

where s is measured from the point of contact where the gear ratio of the twin rolling curve pair is $u=-1$.

Condition 3

$$u_T\left[C+\int_0^{\Theta_L}\frac{1}{u_T(\Theta_L)}\,d\Theta_L\right]=\frac{1}{u_T(\Theta_L)} \quad \ldots\ldots (16)$$

where C = an arbitrary constant

$u_T=u_T(\Theta)$ = the instantaneous gear ratio of the twin rolling curve pair, i.e. the gear ratio when the rolling curve in question meshes with an identical mating rolling curve

C_T = the angle between the centres of rotation of the twin rolling curve pair with appropriate sign

This condition implies that, if we have a point on the gear-ratio curve with the coordinates Θ_L and u_T, then we have on the same wheel a corresponding point with the coordinates

$$\left.\begin{aligned}\Theta_R&=C+\int_0^{\Theta_L}\frac{1}{u_T(\Theta_L)}\,d\Theta_L\\ u_T(\Theta_R)&=\frac{1}{u_T(\Theta_L)}\end{aligned}\right\} \quad \ldots\ldots (17)$$

Θ_L and Θ_R thus here denote polar angles on the same wheel.

Condition 4

If the polar axis is so placed that $\Theta_L=0$ for $u_T=-1$, eqs. (17) take the form

$$\left.\begin{aligned}\Theta_R&=\int_0^{\Theta_L}\frac{1}{u_T(\Theta_L)}\,d\Theta_L\\ u_T(\Theta_R)&=\frac{1}{u_T(\Theta_L)}\end{aligned}\right\} \quad \ldots\ldots (18)$$

If instead the polar axis is so placed that

$$-\int_0^{\frac{180^\circ}{n}} \frac{1}{u_T(\Theta)}\, d\Theta = \frac{180^\circ}{n} \qquad \text{(19)}$$

then eqs. (17) become

$$\left.\begin{aligned} \Theta_R &= \frac{180^\circ}{n} + \int_0^{\Theta_L} \frac{1}{u_T(\Theta_L)}\, d\Theta_L \\ u_T(\Theta_R) &= \frac{1}{u_T(\Theta_L)} \end{aligned}\right\} \qquad \text{(20)}$$

where n is the periodic number.

The latter choice of the position of the polar axis implies that, if the one wheel rotates from $\Theta = 0$ to $\Theta = \frac{180^\circ}{n}$, the mating wheel will rotate through an angle of equal size.

Similarly as for plane rolling curves, the mating curve to a spherical twin rolling curve is itself a spherical twin rolling curve.

Mirror rolling curves are also of certain importance for cylindrical gears, since corresponding wheels can then be made identical, although one of the wheels is turned over to the mirror position in the gear. Since the corresponding possibility does not exist for bevel gears, mirror rolling curves will not be further considered.

CHAPTER III

SPHERICAL COORDINATE SYSTEMS AND THEIR TRANSFORMATION FORMULAE

A. Coordinate systems used

In this investigation two different spherical coordinate systems are used, namely:

a) *Common spherical polar coordinates.* The position of a point in this coordinate system is given by two coordinates: the polar angle and the radius vector. The polar angle is the angle between the polar axis (a great circle) with a given positive direction and the radius vector (geodetic vector). The radius vector is measured as the centre angle on the sphere.

b) A special system of coordinates, named *great circle coordinates*, which is built up in the following manner. Through the north and south poles of the sphere which are denoted by N. and S. we draw great circles (meridians) at constant angular distances from each other. These great circles constitute the x-coordinates. On the equator we imagine a west pole W. and an east pole E. diametrically situated with respect to each other. Through the W. and E. poles we draw similar great circles at constant angular distances. These great circles constitute the y-coordinates. The origin O is placed on the equator equidistant from the W. and E. poles. The equator is considered as the x-axis and the meridian through the origin is considered as the y-axis. Fig. 2 illustrates the construction of the system of great circle coordinates.

B. Transformation between great circle and polar coordinates

The angular distance of a point (x, y) from the x-axis is denoted by Y and from the y-axis by X. The radius vector to the point is denoted by r (see Fig. 2). If we imagine a tangential plane at the origin of the sphere, and if we imagine the radius vector plane and the planes through the coordinate circles x and y extended until they intersect the tangential plane, we obtain the distance from the origin to the intersection with the x-plane equal to $\tan y$ and to the intersection with the y-plane equal to $\tan x$, and further the central projection of the radius vector from the centre of the sphere equal to $\tan r$. The lengths of the distances X and Y are determined by

$$\left.\begin{aligned}\tan X &= \tan x \cos y\\ \tan Y &= \tan y \cos x\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

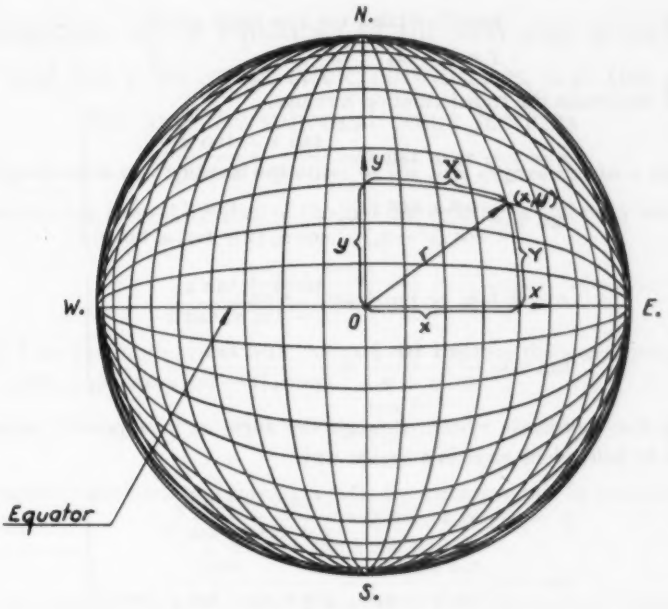


Fig. 2. Great-circle coordinates.

The length of the radius vector r and the polar angle Θ are determined by

$$\left.\begin{aligned} \tan r &= \sqrt{\tan^2 x + \tan^2 y} \\ \tan \Theta &= \frac{\tan y}{\tan x} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

or

$$\left.\begin{aligned} \tan x &= \cos \Theta \tan r \\ \tan y &= \sin \Theta \tan r \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

These systems of equations constitute the transformation formulae between great circle and polar coordinates.

C. Transformation of great circle coordinates

a) Rotation about the axes through the N. and S. and through the W. and E. poles

Such kinds of rotation correspond to a translation along the x- or the y-axis in the case of rectangular coordinates in the plane. If the coordinate system is rotated through the angle α_x in a positive direction about the N. and S. poles alone, the length Y remains unchanged and the x-coordinates are decreased by the angle α_x. If the original coordinates are denoted by x and y and the new coordinates by $\bar{x}$ and $\bar{y}$, we obtain by using eqs. (21)

$$\left.\begin{aligned} &\tan Y = \tan y \cos x = \tan \bar{y} \cos \bar{x} \\ &\bar{x} = x - \alpha_x \end{aligned}\right\}$$

from which we obtain the transformation formulae

$$\left.\begin{aligned} &\bar{x} = x - \alpha_x \text{ or } \tan \bar{x} = \frac{\tan x - \tan \alpha_x}{1 + \tan \alpha_x \tan x} \\ &\tan \bar{y} = \frac{\cos x \tan y}{\cos (x - \alpha_x)} = \frac{\tan y}{\cos \alpha_x (1 + \tan \alpha_x \tan x)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (24)$$

or

$$\left.\begin{aligned} &x = \bar{x} + \alpha_x \text{ or } \tan x = \frac{\tan \bar{x} + \tan \alpha_x}{1 - \tan \alpha_x \tan \bar{x}} \\ &\tan y = \frac{\cos \bar{x} \tan \bar{y}}{\cos (\bar{x} + \alpha_x)} = \frac{\tan \bar{y}}{\cos \alpha_x (1 - \tan \alpha_x \tan \bar{x})} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (25)$$

Rotating the coordinate system through the angle α_y in a positive direction about the W. and E. poles alone gives in a similar way

$$\left.\begin{aligned} &\bar{y} = y - \alpha_y \text{ or } \tan \bar{y} = \frac{\tan y - \tan \alpha_y}{1 + \tan \alpha_y \tan y} \\ &\tan \bar{x} = \frac{\tan x \cos y}{\cos (y - \alpha_y)} = \frac{\tan y}{\cos \alpha_y (1 + \tan \alpha_y \tan y)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (26)$$

or

$$\left.\begin{aligned} &y = \bar{y} + \alpha_y \text{ or } \tan y = \frac{\tan \bar{y} + \tan \alpha_y}{1 - \tan \alpha_y \tan \bar{y}} \\ &\tan x = \frac{\tan \bar{x} \cos \bar{y}}{\cos (\bar{y} + \alpha_y)} = \frac{\tan \bar{x}}{\cos \alpha_y (1 - \tan \alpha_y \tan \bar{y})} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (27)$$

b) Rotation of the coordinate system about the origin

If the coordinate system is rotated through an angle α_r in the positive direction about the origin, then r remains unchanged and Θ decreases the angle α_r. Thus we obtain from eqs. (22)

$$\left.\begin{aligned} &\tan r = \sqrt{\tan^2 \bar{x} + \tan^2 \bar{y}} = \sqrt{\tan^2 x + \tan^2 y} \\ &\tan \bar{\vartheta} = \frac{\tan \bar{y}}{\tan \bar{x}} = \tan (\vartheta - \alpha_r) = \frac{\tan \vartheta - \tan \alpha_r}{1 + \tan \alpha_r \tan \vartheta} \\ &\tan \vartheta = \frac{\tan y}{\tan x} \end{aligned}\right\}$$

from which we obtain the transformation formulae

$$\left.\begin{aligned} &\tan \bar{x} = \cos \alpha_r \tan x + \sin \alpha_r \tan y \\ &\tan \bar{y} = \cos \alpha_r \tan y - \sin \alpha_r \tan x \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (28)$$

or

$$\left.\begin{aligned} &\tan x = \cos \alpha_r \tan \bar{x} - \sin \alpha_r \tan \bar{y} \\ &\tan y = \cos \alpha_r \tan \bar{y} + \sin \alpha_r \tan \bar{x} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (29)$$

where $\bar{x}$ and $\bar{y}$ are the new coordinates, and x and y the original coordinates.

D. Investigation of the equations of the first and second degree in tan x and tan y for great circle coordinates, i. e. the geodetic line and the spherical conic sections

a) The significance of a general equation of the first degree in tan x and tan y

Assume, as given, a general equation of the first degree in tan x and tan y which may be written as

$$\frac{\tan x}{\tan a} + \frac{\tan y}{\tan b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

where a and b are the intersections with the axes. By rotating the axes about the origin through an angle α_r given by

$$\tan \alpha_r = -\frac{\tan b}{\tan a}$$

and by using the transformation formulae (29), the equation can be transformed, after simplification, to

$$\tan \bar{y} = \tan b \cos \alpha_r$$

This equation constitutes a great circle or a geodetic line. The analogy between eq. (30) and the equation of the first degree in the analytic geometry in the plane should be observed. This latter equation can be written as

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the intersections with the axes, and which equation constitutes a straight line. If in the last equation we replace all magnitudes by their tan values, we return to eq. (30).

If we place a tangential plane at the origin of the sphere and project on to this plane the great circle coordinate system of the sphere by means of projectors from the centre of the sphere, we obtain on the tangential plane a system of straight lines intersecting each other at right angles, and having the distances from the projections of the x- and y-axes proportional to tan x and tan y. Accordingly we can obtain the central projection of any arbitrary spherical curve on this tangential plane, if all coordinate values in the equation of the curve in great circle coordinates are replaced by their arc tan values.

b) The significance of a general equation of the second degree in tan x and tan y

Assume, as given, a general equation of the second degree in tan x and tan y in great circle coordinates

$$c_1 \tan^2 x + c_2 \tan^2 y + c_3 \tan x \tan y + c_4 \tan x + c_5 \tan y + c_6 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (31)$$

where c_1, c_2, c_3, c_4, c_5, c_6 are constants.

Firstly, the coordinate system is rotated through an angle α_x about the N. and S. poles by using the transformation formulae (25). We then obtain

$$\left.\begin{aligned}
&\bar{c}_1 \tan^2 \bar{x} + \bar{c}_2 \tan^2 \bar{y} + \bar{c}_3 \tan \bar{x} \tan \bar{y} + \bar{c}_4 \tan \bar{x} + \bar{c}_5 \tan \bar{y} + \bar{c}_6 = 0 \\
&\text{where} \\
&\bar{c}_1 = c_1 - c_4 \tan \alpha_x + c_6 \tan^2 \alpha_x \\
&\bar{c}_2 = \frac{c_2}{\cos^2 \alpha_x} = c_2 (1 + \tan^2 \alpha_x) \\
&\bar{c}_3 = \frac{c_3}{\cos \alpha_x} - c_5 \frac{\tan \alpha_x}{\cos \alpha_x} = \frac{1}{\cos \alpha_x} (c_3 - c_5 \tan \alpha_x) \\
&\bar{c}_4 = 2 (c_1 - c_6) \tan \alpha_x + c_4 (1 - \tan^2 \alpha_x) \\
&\bar{c}_5 = c_3 \frac{\tan \alpha_x}{\cos \alpha_x} + \frac{c_5}{\cos \alpha_x} = \frac{1}{\cos \alpha_x} (c_5 + c_3 \tan \alpha_x) \\
&\bar{c}_6 = c_1 \tan^2 \alpha_x + c_4 \tan \alpha_x + c_6
\end{aligned}\right\} \ldots (32)$$

After that, the coordinate system is rotated through an angle α_y about the W. and E. poles by using the transformation formulae (27). We then obtain, after simplification, and if $\bar{\bar{x}}$ and $\bar{\bar{y}}$ denote the new coordinates,

$$\left.\begin{aligned}
&\bar{\bar{c}}_1 \tan^2 \bar{\bar{x}} + \bar{\bar{c}}_2 \tan^2 \bar{\bar{y}} + \bar{\bar{c}}_3 \tan \bar{\bar{x}} \tan \bar{\bar{y}} + \bar{\bar{c}}_4 \tan \bar{\bar{x}} + \bar{\bar{c}}_5 \tan \bar{\bar{y}} + \bar{\bar{c}}_6 = 0 \\
&\text{where} \\
&\bar{\bar{c}}_1 = \frac{\bar{c}_1}{\cos^2 \alpha_y} \\
&\bar{\bar{c}}_2 = \bar{c}_2 - \bar{c}_5 \tan \alpha_y + \bar{c}_6 \tan^2 \alpha_y \\
&\bar{\bar{c}}_3 = \frac{\bar{c}_3}{\cos \alpha_y} - \bar{c}_4 \frac{\tan \alpha_y}{\cos \alpha_y} \\
&\bar{\bar{c}}_4 = \bar{c}_3 \frac{\tan \alpha_y}{\cos \alpha_y} + \frac{\bar{c}_4}{\cos \alpha_y} \\
&\bar{\bar{c}}_5 = 2 \bar{c}_2 \tan \alpha_y + \bar{c}_5 (1 - \tan^2 \alpha_y) - 2 \bar{c}_6 \tan \alpha_y \\
&\bar{\bar{c}}_6 = \bar{c}_2 \tan^2 \alpha_y + \bar{c}_5 \tan \alpha_y + \bar{c}_6
\end{aligned}\right\} \ldots (33)$$

By choosing suitable values of α_x and α_y, the terms of the first degree in $\tan \bar{\bar{x}}$ and $\tan \bar{\bar{y}}$ can be eliminated. We then have

$$\left.\begin{aligned}
&\bar{\bar{c}}_4 = \bar{c}_3 \frac{\tan \alpha_y}{\cos \alpha_y} + \frac{\bar{c}_4}{\cos \alpha_y} = 0 \\
&\bar{\bar{c}}_5 = 2 \bar{c}_2 \tan \alpha_y + \bar{c}_5 (1 - \tan^2 \alpha_y) - 2 \bar{c}_6 \tan \alpha_y = 0
\end{aligned}\right\}$$

or

$$\left.\begin{aligned}
&\tan \alpha_y = -\frac{\bar{c}_4}{\bar{c}_3} = \frac{c_4 \tan^2 \alpha_x + 2 (c_1 - c_2) \tan \alpha_x - c_4}{c_3 - c_5 \tan \alpha_x} \cos \alpha_x \\
&\tan 2 \alpha_y = -\frac{\bar{c}_5}{\bar{c}_2 - \bar{c}_6}
\end{aligned}\right\} \ldots\ldots\ldots (34)$$

If we determine $\tan 2\alpha_y$ from the first equation in (34) and substitute in the second one, we obtain

$$\frac{\bar{c}_3}{\bar{c}_4} + \frac{2\bar{c}_6}{\bar{c}_5} = \frac{\bar{c}_4}{\bar{c}_3} + \frac{2\bar{c}_2}{\bar{c}_5}$$

If the expressions for $\bar{c}_2$, $\bar{c}_3$, $\bar{c}_4$, $\bar{c}_5$, and $\bar{c}_6$ from eqs. (32) are substituted in the last equation, we obtain

$$\frac{(c_3 - c_5 \tan \alpha_x)(1 + \tan^2 \alpha_x)}{2(c_1 - c_6)\tan \alpha_x + c_4(1 - \tan^2 \alpha_x)} + \frac{2(c_1 \tan^2 \alpha_x + c_4 \tan \alpha_x + c_6)}{c_5 + c_3 \tan \alpha_x} =$$
$$= \frac{2(c_1 - c_6)\tan \alpha_x + c_4(1 - \tan^2 \alpha_x)}{c_3 - c_5 \tan \alpha_x} + \frac{2c_2(1 + \tan^2 \alpha_x)}{c_5 + c_3 \tan \alpha_x}$$

If we multiply this equation by the denominators and divide through by $1 + \tan^2 \alpha_x$, we obtain, after simplification,

$$\left.\begin{aligned} &\tan^3 \alpha_x + C_2 \tan^2 \alpha_x + C_4 \tan \alpha_x + C_6 = 0 \\ &\text{where} \\ &C_2 = \frac{c_5(4c_1^2 + 2c_3^2 - c_4^2 - c_5^2) - 2c_3c_4(c_1 + c_2 - 2c_6) - 4c_1c_5c_6 + 4c_2c_5(c_6 - c_1)}{c_3(c_4^2 - c_5^2) - 2c_4c_5(c_1 - c_2)} \\ &C_4 = \frac{c_3(2c_5^2 + 4c_6^2 - c_3^2 - c_4^2) + 2c_4c_5(2c_1 - c_2 - c_6) - 4c_2c_3c_6 + 4c_1c_3(c_2 - c_6)}{c_3(c_4^2 - c_5^2) - 2c_4c_5(c_1 - c_2)} \\ &C_6 = \frac{c_5(c_4^2 - c_3^2) + 2c_3c_4(c_6 - c_2)}{c_3(c_4^2 - c_5^2) - 2c_4c_5(c_1 - c_2)} \end{aligned}\right\} \quad (35)$$

As is the case with all equations of the third degree, this equation must have at least one real root. In fact, all three roots are real as will be shown later. The equation, therefore, is the irreducible case. Thus, if the coordinate system is firstly rotated through an angle α_x about the N. and S. poles given by eqs. (35) and secondly through an angle α_y about the W. and E. poles determined by the first equation in the system of equations (34), the equation of the curve will take the form

$$\bar{\bar{c}}_1 \tan^2 \bar{\bar{x}} + \bar{\bar{c}}_2 \tan^2 \bar{\bar{y}} + \bar{\bar{c}}_3 \tan \bar{\bar{x}} \tan \bar{\bar{y}} + \bar{\bar{c}}_6 = 0$$

Finally, we rotate the coordinate system through an angle α_r about the origin, using the transformation formulae (29) and choosing α_r from

$$\tan 2\alpha_r = \frac{\bar{\bar{c}}_3}{\bar{\bar{c}}_1 - \bar{\bar{c}}_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

The equation then takes the form

$$\left.\begin{aligned} &\bar{\bar{c}}_1 \tan^2 \bar{\bar{\bar{x}}} + \bar{\bar{\bar{c}}}_2 \tan^2 \bar{\bar{\bar{y}}} + \bar{\bar{\bar{c}}}_6 = 0 \\ &\text{where} \\ &\bar{\bar{\bar{c}}}_1 = \bar{\bar{c}}_1 \cos^2 \alpha_r + \bar{\bar{c}}_2 \sin^2 \alpha_r + \bar{\bar{c}}_3 \sin \alpha_r \cos \alpha_r \\ &\bar{\bar{\bar{c}}}_2 = \bar{\bar{c}}_1 \sin^2 \alpha_r + \bar{\bar{c}}_2 \cos^2 \alpha_r - \bar{\bar{c}}_3 \sin \alpha_r \cos \alpha_r \\ &\bar{\bar{\bar{c}}}_6 = \bar{\bar{c}}_6 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (37)$$

Eq. (36) gives four different solutions for α_r which differ 90° from each other. From the second and the third equation in (37), it can be seen that for two different values of α_r, differing 90° from each other, the values of $\bar{\bar{\bar{c}}}_1$ and $\bar{\bar{\bar{c}}}_2$ are reversed. It is therefore always possible to choose a value of α_r from eq. (36), as follows.

If $\bar{\bar{\bar{c}}}_1$ and $\bar{\bar{\bar{c}}}_2$ have the same signs, and $\bar{\bar{\bar{c}}}_6$ opposite sign, we can obtain $|\bar{\bar{\bar{c}}}_1| \leq |\bar{\bar{\bar{c}}}_2|$.

If all values of $\bar{\bar{c}}$ have the same signs, we can have either

$$|\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_2| \text{ or } |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_2| \text{ or } |\bar{\bar{c}}_2| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6|.$$

If $\bar{\bar{c}}_1$ and $\bar{\bar{c}}_2$ have opposite signs, we can choose α_r either so that $|\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_2|$ and that $\bar{\bar{c}}_6$ has the opposite sign to $\bar{\bar{c}}_1$, or that $|\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6|$ and $\bar{\bar{c}}_1$ and $\bar{\bar{c}}_6$ have the same sign.

If α_r is chosen according to these conditions, eqs. (37) can be written in one of the following forms

$$\left.\begin{aligned}
&|\bar{\bar{c}}_1| \tan^2 \bar{\bar{x}} + |\bar{\bar{c}}_2| \tan^2 \bar{\bar{y}} - |\bar{\bar{c}}_6| = 0 \text{ where } |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_2| \\
&|\bar{\bar{c}}_1| \tan^2 \bar{\bar{x}} + |\bar{\bar{c}}_2| \tan^2 \bar{\bar{y}} + |\bar{\bar{c}}_6| = 0 \text{ where either } |\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_2| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or } |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_2| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or } |\bar{\bar{c}}_2| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6| \\
&|\bar{\bar{c}}_1| \tan^2 \bar{\bar{x}} - |\bar{\bar{c}}_2| \tan^2 \bar{\bar{y}} \mp |\bar{\bar{c}}_6| = 0 \text{ where } |\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_2| \text{ for the } - \text{ sign before } |\bar{\bar{c}}_6| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or } |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6| \text{ for the } + \text{ sign before } |\bar{\bar{c}}_6|
\end{aligned}\right\} \quad (38)$$

If in eqs. (38) $\tan \bar{\bar{x}}$ is replaced by x, and $\tan \bar{\bar{y}}$ is replaced by y, we obtain the following analogies to the plane curves:

The first equation is analogous to the equation of the ellipse

The second equation is analogous to the equation of the ellipse with imaginary axes

The third equation is analogous to the equation of the hyperbola.

According to these analogies, we could name the different curves: the spherical ellipse, the spherical ellipse with imaginary axes, and the spherical hyperbola. However, as will be seen from the following, the spherical hyperbola is not an independent curve, but it can be transformed by means of coordinate transformations into a spherical ellipse.

Eqs. (38) will now be further transformed in the following way. If we introduce the substitutions

$$\tan^2 a = \left|\frac{\bar{\bar{c}}_6}{\bar{\bar{c}}_1}\right| \text{ and } \tan^2 b = \left|\frac{\bar{\bar{c}}_6}{\bar{\bar{c}}_2}\right|$$

the first equation can be rewritten, if the bars are omitted, in the form

$$\left.\begin{aligned}
&\frac{\tan^2 x}{\tan^2 a} + \frac{\tan^2 y}{\tan^2 b} = 1 \text{ or } \frac{\tan^2 x}{\tan^2 a} + \frac{\tan^2 y}{\left(\frac{\cos ea}{\cos a}\right)^2 - 1} = 1 \\
&\text{where} \\
&\cos b = \frac{\cos a}{\cos ea} \\
&\text{and } 0 \leq a \leq 90^\circ,\ 0 \leq e \leq 1 \text{ and } 0 \leq b \leq a
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (39)$$

The corresponding curve is called the *real spherical conic section* (or the *spherical conic section with real axes*). If $y = 0$, then $x = a$; and if $x = 0$, then $y = b$; thus a is half the major axis and b half the minor axis of the curve; e is the eccentricity of the curve. The points given by $\tan y = 0$, $\tan x = \tan (e\,a)$ and by $\tan y = 0$, $\tan x = -\tan (e\,a)$ are the foci.

If, in the second equation of (38) $|\bar{\bar{c}}_0| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_2|$
we can then introduce the substitutions

$$\left[\frac{1}{i}\tan i\,|a|\right]^2 = \tanh^2|a| = \left|\frac{\bar{\bar{c}}_0}{\bar{\bar{c}}_1}\right| \text{ and } \left[\frac{1}{i}\tan i\,|b|\right]^2 = \tanh^2|b| = \left|\frac{\bar{\bar{c}}_0}{\bar{\bar{c}}_2}\right|$$

where i is the unit of imaginaries.

If, on the contrary, $|\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_0| \leq |\bar{\bar{c}}_2|$ then firstly the coordinate system is rotated through 90° about the N. and S. poles by means of the transformation formulae (25). The equation thereafter takes the form

$$|\bar{\bar{c}}_0|\tan^2\bar{\bar{x}} + |\bar{\bar{c}}_2|\tan^2\bar{\bar{y}} + |\bar{\bar{c}}_1| = 0$$

After that we introduce the substitutions

$$\left[\frac{1}{i}\tan i\,|a|\right]^2 = \tanh^2|a| = \left|\frac{\bar{\bar{c}}_1}{\bar{\bar{c}}_0}\right| \text{ and } \left[\frac{1}{i}\tan i\,|b|\right]^2 = \tanh^2|b| = \left|\frac{\bar{\bar{c}}_1}{\bar{\bar{c}}_2}\right|$$

If, finally, $|\bar{\bar{c}}_2| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_0|$ the coordinate system is rotated through 90° about the W. and E. poles by means of the transformation formulae (27), the equation then takes the form

$$|\bar{\bar{c}}_1|\tan^2\bar{\bar{x}} + |\bar{\bar{c}}_0|\tan^2\bar{\bar{y}} + |\bar{\bar{c}}_2| = 0$$

After that we introduce the substitutions

$$\left[\frac{1}{i}\tan i\,|a|\right]^2 = \tanh^2|a| = \left|\frac{\bar{\bar{c}}_2}{\bar{\bar{c}}_1}\right| \text{ and } \left[\frac{1}{i}\tan i\,|b|\right]^2 = \tanh^2|b| = \left|\frac{\bar{\bar{c}}_2}{\bar{\bar{c}}_0}\right|$$

In all three cases, therefore, the second equation in (38) can be written, if the bars are omitted, in the form

$$\left.\begin{aligned} &\frac{\tan^2 x}{\tan^2 i\,|a|} + \frac{\tan^2 y}{\tan^2 i\,|b|} = 1 \text{ or } \frac{\tan^2 x}{\tanh^2|a|} + \frac{\tan^2 y}{\tanh^2|b|} = -1 \\ &\text{where} \\ &\cos i\,|b| = \frac{\cos i\,|a|}{\cos ei\,|a|} \text{ or } \cosh|b| = \frac{\cosh|a|}{\cos ei\,|a|} \end{aligned}\right\} \quad \ldots\ldots\ldots (40)$$

where e is real and <1, or imaginary.

The corresponding curve is named the *spherical conic section with imaginary axes* or the *imaginary spherical conic section.* If e is real and < 1, we call the curve the *imaginary spherical ellipse;* and if e is imaginary, the curve is called the *imaginary spherical hyperbola.* This curve is not a curve in the real spherical surface.

If in the third equation of (38), the negative sign is valid for $|\bar{\bar{c}}_0|$ the coordinate system is rotated through 90° about the N. and S. poles and we obtain

$$-|\bar{\bar{c}}_0|\tan^2\bar{\bar{x}} - |\bar{\bar{c}}_2|\tan^2\bar{\bar{y}} + |\bar{\bar{c}}_1| = 0$$

We then introduce

$$\tan^2 a = \left|\frac{\bar{\bar{c}}_1}{\bar{\bar{c}}_0}\right| \text{ and } \tan^2 b = \left|\frac{\bar{\bar{c}}_1}{\bar{\bar{c}}_2}\right|$$

If on the contrary the positive sign is valid for $\bar{\bar{c}}_6$, the coordinate system is rotated 90° about the W. and E. poles, and the equation takes the form

$$|\bar{\bar{c}}_1| \tan^2 \bar{\bar{x}} + |\bar{\bar{c}}_6| \tan^2 \bar{\bar{y}} - |\bar{\bar{c}}_2| = 0$$

We then introduce

$$\tan^2 a = \left|\frac{\bar{\bar{c}}_2}{\bar{\bar{c}}_1}\right| \text{ and } \tan^2 b = \left|\frac{\bar{\bar{c}}_2}{\bar{\bar{c}}_1}\right|$$

In both cases the equation can now be written in the form (39). The spherical hyperbola therefore does not exist as an independent curve, neither does there exist any independent spherical parabola; this can also be understood from the fact that the plane parabola is a limiting case for the ellipse, when $a \to \infty$, but for the spherical ellipse we always have $a \leq 90°$. By means of coordinate transformations, however, the spherical conic section curve according to eqs. (39), can be brought to forms analogous to the plane hyperbola and parabola. If the coordinate system is rotated through 90° about the N. and S. poles, the eqs. (39) take the following form, similar to the equation for the plane hyperbola

$$\left.\begin{aligned} &\frac{\tan^2 x}{\tan^2 \bar{a}} - \frac{\tan^2 y}{\tan^2 \bar{b}} = 1 \text{ or } \frac{\tan^2 x}{\tan^2 \bar{a}} - \frac{\tan^2 y}{\left(\frac{\cos \bar{e}\,\bar{a}}{\cos \bar{a}}\right)^2 - 1} = 1 \\ &\text{where} \\ &\bar{a} = 90° - a \\ &\bar{e}\,\bar{a} = 90° - e\,a \\ &\cos \bar{b} = \frac{\cos \bar{a}}{\cos \bar{e}\,\bar{a}} \end{aligned}\right\} \quad \ldots\ldots\ldots (41)$$

If instead, the coordinate system is rotated through the angle $a - 90°$ about the N. and S. poles we obtain the following equation, similar to the equation for the plane parabola

$$\tan^2 y = 2 \frac{\tan^2 b}{\tan a} [\tan x - \tan (90° - 2\,a)] \quad \ldots\ldots\ldots\ldots\ldots (42)$$

Since tan x and tan y each correspond to two different values of x and y, differing 180° from one another, eqs. (39) give two closed curves, one on the front side of the sphere, i.e. the hemisphere which has the origin as midpoint, and another identical curve on the rear side. The first curve is named the *main curve*, and the second curve is named the *antipode curve*. There are also two pairs of foci, one pair lying inside the main curve, and the other pair (the antipode foci) inside the antipode curve.

It is now easy to understand the geometrical meaning of the three roots to eq. (35), and it can be seen that they must all three be real. To enable the term of the first degree in tan x and tan y to disappear, it is evident that we can choose three different values of tan α_x; firstly tan α_x can be chosen so that the midpoint of the closed curve, after rotation about the N. and S. poles, falls on the y-axis, and then, after rotation about the W. and E. poles, falls at the origin. This choice corresponds to the first equation in (38), or to the second equation with $|\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_2|$. Secondly we can choose tan α_x in such a way that a point

lying either on the extension of the major or the minor axis and half way between the midpoints of the two closed curves falls on the y-axis after the first rotation of the coordinate system, and falls in the origin after the second rotation. These two cases correspond to the third equation in (38), or to the second equation, if $|\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6| \leq |\bar{\bar{c}}_2|$ or $|\bar{\bar{c}}_2| \leq |\bar{\bar{c}}_1| \leq |\bar{\bar{c}}_6|$.

Eq. (35) therefore always must have three real solutions for $\tan \alpha_x$. Two of the roots, however, may be double roots. If initially a suitable root is chosen, and if we choose a suitable rotation about the W. and E. poles, and about the origin, we can obtain in all three cases eqs. (38) in such a form that further coordinate transformations are not necessary to bring the original equation of the curve into the forms (39) or (40).

Since an equation of the second degree in $\tan x$ and $\tan y$ remains a second degree equation in these variables independently of the rotations of the coordinate axes, it can be understood that if we place a tangential plane at an arbitrary point on the sphere, and project radially the curve on to this plane, the projected curve will always become a plane conic section. According to where the plane is placed the projected curve becomes an ellipse, a hyperbola or a parabola.

Finally, it will be shown that the spherical conic section curve has the same property as the plane ellipse, namely that the sum of the distances from an arbitrary point on the curve to the two foci, is constant and equals $2a$.

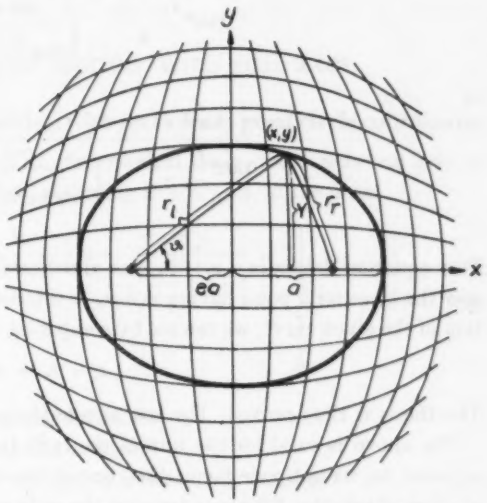


Fig. 3. From the spherical right-angled triangle $(x + ea)$, Y, r_l we have $\tan \vartheta = \dfrac{\tan Y}{\sin (x + ea)}$.

We form at first an expression for the two focal radii which are denoted by r_l and r_r (see Fig. 3). A known formula for a right-angled spherical triangle states

$$\left.\begin{aligned} \cos r_l &= \cos Y \cos (x + ea) \\ \cos r_r &= \cos Y \cos (x - ea) \end{aligned}\right\}$$

or

$$\left.\begin{aligned} \frac{\cos r_l + \cos r_r}{2} &= \cos \frac{r_l + r_r}{2} \cos \frac{r_l - r_r}{2} = \cos Y \cos ea \cos x \\ -\frac{\cos r_l - \cos r_r}{2} &= \sin \frac{r_l + r_r}{2} \sin \frac{r_l - r_r}{2} = \cos Y \sin ea \sin x \end{aligned}\right\}$$

or squaring and adding the equations

$$1 = \frac{\cos^2 Y \cos^2 ea \cos^2 x}{\cos^2 \dfrac{r_l + r_r}{2}} + \frac{\cos^2 Y \sin^2 ea \sin^2 x}{\sin^2 \dfrac{r_l + r_r}{2}}$$

From eqs. (21) we obtain

$$\cos^2 Y = \frac{1}{1 + \cos^2 x \tan^2 y} = \frac{1 + \tan^2 x}{1 + \tan^2 x + \tan^2 y}$$

If this expression and also

$\cos^2 x = \dfrac{1}{1 + \tan^2 x}$, $\sin^2 x = \dfrac{\tan^2 x}{1 + \tan^2 x}$ are inserted, we obtain after simplification

$$\frac{\tan^2 x}{\tan^2 \dfrac{r_l + r_r}{2}} + \frac{\tan^2 y}{\left(\dfrac{\cos ea}{\cos \dfrac{r_l + r_r}{2}}\right)^2 - 1} = 1$$

or

$$\frac{\tan^2 x}{\tan^2 \dfrac{r_l + r_r}{2}} + \frac{\tan^2 y}{\left(\dfrac{\cos a}{\cos b \cos \dfrac{r_l + r_r}{2}}\right)^2 - 1} = 1$$

This equation determines $r_l + r_r$, if x and y as well as a and e are given. The equation is also the locus for a point having $r_l + r_r =$ a constant. Since x and y must satisfy the equation of the given curve, we can see by comparing eqs. (39) and (40) that we obtain

$$r_l + r_r = 2a$$

The thesis is thus proved. The law also holds good if a and b are imaginary.

The above is valid for the sum of the radii to the main foci. If one of the main foci is replaced by its antipode focus, then one of the radii, e.g. r_l, is changed to $180° - r_l$, and we then obtain the difference between the radii equal to the constant $180° - 2a$, similarly as for the plane hyperbola.

The name *spherical conic sections* used for curves of the second degree in $\tan x$ and $\tan y$ in great circle coordinates, is perhaps not quite adequate. Plane conic sections are obtained as the intersection between a plane and a circular cone. A spherical conic section, however, cannot generally be obtained as the intersection between a circular cone and a sphere. It can, however, be obtained as the intersection between a sphere and a cone having a plane conic section as base, and having its apex in the centre of the sphere, which to some degree justifies the name spherical conic section.

CHAPTER IV

EQUATIONS AND ARC LENGTHS OF SPHERICAL CONIC SECTION CURVES

A. The spherical conic section with real axes

a) The equation of the spherical conic section with real axes in great circle coordinates

In great circle coordinates the equation of the *spherical conic section with real axes* or the *real spherical conic section* can be written, according to eqs. (39), in the form

$$\left.\begin{aligned} &\frac{\tan^2 x}{\tan^2 a}+\frac{\tan^2 y}{\tan^2 b}=1 \text{ or } \frac{\tan^2 x}{\tan^2 a}+\frac{\tan^2 y}{\left(\frac{\cos ea}{\cos a}\right)^2-1}=1 \\ \text{where} \quad &\cos b=\frac{\cos a}{\cos ea} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (43)$$

and where a = half the major axis (in angular measure)
b = half the minor axis (in angular measure)
e = the eccentricity ($e<1$ for the real spherical conic section)

Introducing the substitution

$$\frac{\tan x}{\tan a}=\sin \varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (44)$$

the equation may be written in the parametric form

$$\left.\begin{aligned} \tan x &= \tan a \sin \varphi \\ \tan y &= \tan b \cos \varphi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

b) The equation of the spherical conic section with real axes in spherical polar coordinates

Eqs. (43) and (45) for the real spherical conic section will now be transformed into spherical polar coordinates with the left-hand focus as pole.

For the spherical right-angled triangle $(x+ea)$, Y, r_1 (see Fig. 3) we have

$$\tan \vartheta=\frac{\tan Y}{\sin (x+ea)}$$

where ϑ = the polar angle.

Substituting from eqs. (21) and (45) we obtain

$$\tan \vartheta = \frac{\cos x \tan b \cos \varphi}{\sin x \cos ea + \cos x \sin ea} = \frac{\tan b \cos \varphi}{\tan a \sin \varphi \cos ea + \sin ea}$$

If, in addition, we introduce the substitution

$$\sin \alpha = \frac{\tan ea}{\tan a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (46)$$

where α = the eccentricity angle, the above expression takes the form

$$\tan \vartheta = \frac{\tan b}{\tan a \cos ea} \cdot \frac{\cos \varphi}{\sin \alpha + \sin \varphi}$$

From eq. (46) and the last equation in (43) we can derive

$$\cos \alpha = \frac{\tan b}{\tan a \cos ea}$$

Substituting this in the above expression for $\tan \vartheta$ we obtain

$$\tan \vartheta = \frac{\cos \alpha \cos \varphi}{\sin \alpha + \sin \varphi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (47)$$

or

$$\cos \vartheta = \frac{\sin \alpha + \sin \varphi}{1 + \sin \alpha \sin \varphi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

and

$$\sin \vartheta = \frac{\cos \alpha \cos \varphi}{1 + \sin \alpha \sin \varphi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (49)$$

From the expressions (48) and (49) for $\cos \vartheta$ and $\sin \vartheta$ we obtain

$$\tan \frac{\vartheta}{2} = \frac{1 - \cos \vartheta}{\sin \vartheta} = \frac{1 - \sin \alpha}{\cos \alpha} \cdot \frac{1 - \sin \varphi}{\cos \varphi}$$

or

$$\tan \frac{\vartheta}{2} = \tan \frac{90^\circ - \alpha}{2} \tan \frac{90^\circ - \varphi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (50)$$

Furthermore, we obtain from the spherical right-angled triangle $(x + ea)$, Y, r_1 in Fig. 3, if the notation r_1 is replaced by r,

$$\tan r = \frac{\tan (x + ea)}{\cos \vartheta}$$

or, if the expressions for $\cos \vartheta$ from (48) and $\tan x$ from (45) and $\sin \alpha$ from (46) are substituted,

$$\frac{\tan r - \tan a}{1 + \tan r \tan a} = \tan a \sin \alpha \sin \varphi$$

or

$$\tan (r - a) = \tan a \sin \alpha \sin \varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (51)$$

The equation for the real spherical conic section in polar coordinates in the parametric form with the parameter φ therefore becomes

$$\left.\begin{aligned} &\tan(r-a) = \tan a \sin\alpha \sin\varphi \\ &\tan\frac{\vartheta}{2} = \tan\frac{90^\circ-\alpha}{2}\tan\frac{90^\circ-\varphi}{2} \\ &\sin\alpha = \frac{\tan ea}{\tan a} \\ &a \leq 90^\circ,\ e \leq 1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52)$$

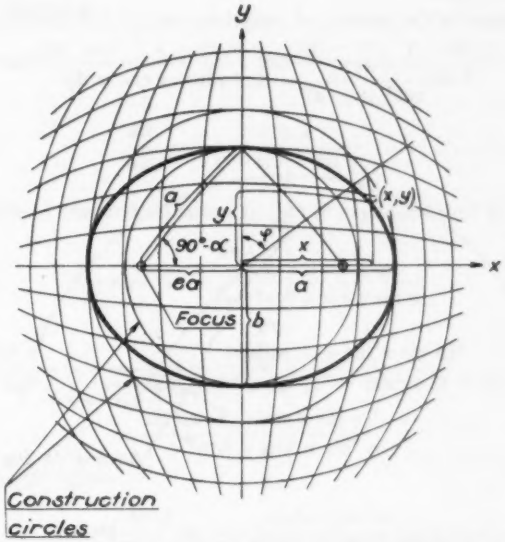


Fig. 4. Construction of a spherical ellipse by means of two construction circles and the parameter angle φ.

To obtain the equation in non-parametric form, we eliminate φ from the system of equations (52). This is easily done by replacing the second equation in the system by the equivalent equation (48). By eliminating φ from eq. (48) and the first equation in (52) we obtain

$$\cos\vartheta = \frac{\tan a \sin^2\alpha + \tan(r-a)}{\sin\alpha\,[\tan a + \tan(r-a)]}$$

or, after expanding $\tan(r-a)$ and solving with respect to $\tan r$,

$$\tan r = \frac{\tan a \cos^2\alpha}{1 + \tan^2 a \sin^2\alpha - \sin\alpha\,(1+\tan^2 a)\cos\vartheta} \quad \ldots\ldots\ldots\ldots \quad (53)$$

which is the equation for the real spherical conic section in polar coordinates in non-parametric form.

Fig. 4 gives the geometrical interpretation of the magnitudes a, b, α and φ.

c) The angle between the radius vector and the normal

Let us consider an infinitesimal arc length ds and draw a triangle with ds as base and with the vertex at the origin. We then obtain (see Fig. 5)

$$\tan \Phi = -\frac{dr}{\sin r \, d\vartheta} = -\frac{\frac{dr}{d\varphi}}{\sin r \frac{d\vartheta}{d\varphi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (54)$$

where Φ = the angle from the radius vector to the normal to the curve, taken with appropriate sign.

If the first equation in the system of equations (52) is differentiated, we obtain

$$\frac{dr}{\cos^2 (r - a)} = \tan a \sin \alpha \cos \varphi \, d\varphi$$

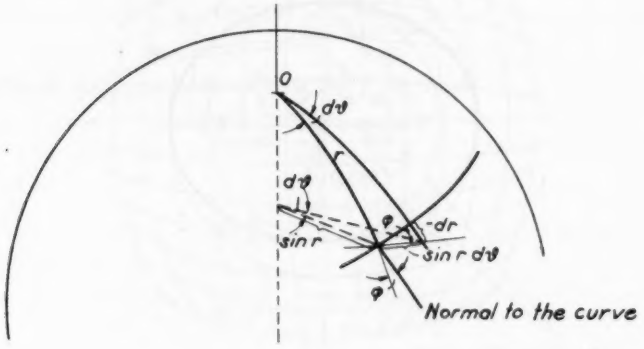


Fig. 5. From the figure we obtain $\tan \Phi = -\frac{dr}{\sin r \, d\vartheta} = -\frac{\frac{dr}{d\varphi}}{\sin r \frac{d\vartheta}{d\varphi}}$

Using the formula $\cos^2 v = \frac{1}{1 + \tan^2 v}$, the same equation may be written as

$$\cos^2 (r - a) = \frac{1}{1 + \tan^2 a \sin^2 \alpha \sin^2 \varphi}$$

which, substituted in the above expression, gives

$$\frac{dr}{d\varphi} = \frac{\tan a \sin \alpha \cos \varphi}{1 + \tan^2 a \sin^2 \alpha \sin^2 \varphi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (55)$$

From the second equation in the system of equations (52) we obtain

$$\frac{d\vartheta}{\cos^2 \frac{\vartheta}{2}} = -\tan \frac{90^\circ - \alpha}{2} \cdot \frac{d\varphi}{\cos^2 \frac{90^\circ - \varphi}{2}}$$

But the same equation may also be rewritten as

$$\frac{1}{\cos^2\frac{\vartheta}{2}} = 1 + \tan^2\frac{90^\circ - \alpha}{2}\tan^2\frac{90^\circ - \varphi}{2}$$

The last two equations give

$$\frac{d\vartheta}{d\varphi} = -\tan\frac{90^\circ - \alpha}{2}\cdot\frac{1 + \tan^2\frac{90^\circ - \varphi}{2}}{1 + \tan^2\frac{90^\circ - \alpha}{2}\tan^2\frac{90^\circ - \varphi}{2}}$$

which, by using the formulae

$$\tan^2\frac{90^\circ - v}{2} = \frac{1 - \sin v}{1 + \sin v},\quad \tan\frac{90^\circ - v}{2} = \frac{1 - \sin v}{\cos v}$$

reduces to

$$\frac{d\vartheta}{d\varphi} = -\frac{\cos\alpha}{1 + \sin\alpha\sin\varphi} \qquad \text{(56)}$$

If the left-hand side of the first equation in (52) is expanded and if we solve for $\tan r$, we obtain

$$\tan r = \tan a\,\frac{1 + \sin\alpha\sin\varphi}{1 - \tan^2 a\sin\alpha\sin\varphi}$$

from which

$$\sin^2 r = \frac{\tan^2 r}{1 + \tan^2 r} = \frac{\tan^2 a\,(1 + \sin\alpha\sin\varphi)^2}{(1 - \tan^2 a\sin\alpha\sin\varphi)^2 + \tan^2 a\,(1 + \sin\alpha\sin\varphi)^2} =$$

$$= \frac{\sin^2 a\,(1 + \sin\alpha\sin\varphi)^2}{1 + \tan^2 a\sin^2\alpha\sin^2\varphi}$$

which, together with (56), gives

$$\sin^2 r\left(\frac{d\vartheta}{d\varphi}\right)^2 = \frac{\sin^2 a\cos^2\alpha}{1 + \tan^2 a\sin^2\alpha\sin^2\varphi} \qquad \text{(57)}$$

As the *main curve*, we define the curve obtained if $(r - a)$ according to eqs. (52) is chosen in the first or fourth quadrant. As the *antipode curve*, we define the curve obtained if $(r - a)$ is chosen in the second or third quadrant.

It will now be shown that $\sin r$ becomes positive for the main curve, and negative for the antipode curve.

For the main curve we have, according to the first and third equations in (52), $r_{max} - a = ea$. Hence $r_{max} = a\,(1 + e)$ and $r_{min} - a = -ea$, and thus $r_{min} = a\,(1 - e)$. Since $a \leq 90^\circ$ and $e \leq 1$, we obtain for the main curve $0 \leq r \leq 180^\circ$, i.e. $\sin r$ is always positive for the main curve.

For the antipode curve we obtain in a similar way $-180^\circ \leq r \leq 0$, i.e. $\sin r$ is always negative for the antipode curve.

From eqs. (54), (55) and (57), since $\frac{d\vartheta}{d\varphi}$ is always negative according to eq. (56), we have

$$\tan\Phi = \pm\frac{\tan\alpha\cos\varphi}{\cos a\sqrt{1 + \tan^2 a\sin^2\alpha\sin^2\varphi}} \qquad \text{(58)}$$

where the + sign is valid if sin r is positive, i.e. for the main curve
" — " " " " " " " " negative, " " " antipode curve

Using the first equation in (52), eq. (58) may be rewritten as

$$\tan \Phi = \frac{\tan \alpha \cos (r - a) \cos \varphi}{\cos a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (59)$$

From eq. (58) it can be seen that tan Φ must have the same sign as cos φ for the main curve, and opposite sign for the antipode curve. This condition will be fulfilled automatically by eq. (59). Therefore no double signs are required here.

d) The arc length of the real spherical conic section curve

The arc length, measured in radians, of a spherical curve is determined by

$$dS = \sqrt{(dr)^2 + \sin^2 r \, (d\vartheta)^2}$$

as may easily be seen from Fig. 5. If for the real spherical ellipse the arc length S is given the suffix e, the above expression may be rewritten as

$$dS_e = \sqrt{\left(\frac{dr}{d\varphi}\right)^2 + \sin^2 r \left(\frac{d\vartheta}{d\varphi}\right)^2} \, d\varphi$$

or substituting $\frac{dr}{d\varphi}$ from eq. (55), and $\sin^2 r \left(\frac{d\vartheta}{d\varphi}\right)^2$ from eq. (57), and after simplification

$$dS_e = \tan a \sqrt{1 - \sin^2 a \cos^2 \alpha} \, \frac{\sqrt{1 - \sin^2 \alpha \sin^2 \varphi}}{1 + \tan^2 a \sin^2 \alpha \sin^2 \varphi} \, d\varphi \quad \ldots\ldots\ldots \quad (60)$$

To integrate this expression we introduce the substitutions

$$\left.\begin{aligned} \Delta\varphi &= \sqrt{1 - \sin^2 \alpha \sin^2 \varphi} \\ -p_1 &= \tan^2 a \sin^2 \alpha \\ &\text{N.B. } p_1 \text{ is negative} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

and eq. (60) then takes the form

$$dS_e = \frac{\tan a \sqrt{1 - \sin^2 a \cos^2 \alpha}}{1 - p_1 \sin^2 \varphi} \cdot \frac{1 - \sin^2 \alpha \sin^2 \varphi}{\Delta\varphi} \, d\varphi$$

or

$$dS_e = \tan a \sqrt{1 - \sin^2 a \cos^2 \alpha} \left[\frac{1 - \dfrac{\sin^2 \alpha}{p_1}}{(1 - p_1 \sin^2 \varphi) \, \Delta\varphi} \, d\varphi + \frac{\dfrac{\sin^2 \alpha}{p_1}}{\Delta\varphi} \, d\varphi \right]$$

or, if p_1 from eq. (61) is substituted in the numerators,

$$dS_e = \frac{\sqrt{1 - \sin^2 a \cos^2 \alpha}}{\sin a \cos a} \cdot \frac{d\varphi}{(1 - p_1 \sin^2 \varphi) \, \Delta\varphi} - \frac{\sqrt{1 - \sin^2 a \cos^2 \alpha}}{\tan a} \cdot \frac{d\varphi}{\Delta\varphi}$$

Integrating, and measuring S_s from the point where $\varphi = 0$,

$$S_s = \sqrt{\frac{1}{\tan^2 a} + \sin^2 \alpha}\left[\frac{1}{\cos a}\, \Pi\,(p_1, \alpha, \varphi) - \cos a \cdot F\,(\alpha, \varphi)\right] \quad \ldots\ldots\ldots \quad (62)$$

where $\Pi\,(p, \alpha, \varphi)$ is the incomplete elliptic integral of the third kind and $F(\alpha, \varphi)$ is the incomplete elliptic integral of the first kind. The corresponding complete integrals are denoted by $\Pi(p, \alpha)$ and $F(\alpha)$. In the following we also use the notations $E(\alpha, \varphi)$ for the incomplete and $E(\alpha)$ for the complete elliptic integral of the second kind.

An integral of the third kind having a negative value of the parameter can, however, be transformed into an integral having a positive value of the parameter p, where $\sin^2\alpha < p < 1$. For this purpose the expression (62) is transformed as follows. According to eq. (61) we have $\tan^2 a = -\dfrac{p_1}{\sin^2 \alpha}$ from which $\cos^2 a = \dfrac{1}{1 - \dfrac{p_1}{\sin^2 \alpha}} = \dfrac{\sin^2 \alpha}{\sin^2 \alpha - p_1}$, and if we further introduce the substitution

$$p = \frac{\sin^2 \alpha}{1 - \sin^2 a \, \cos^2 \alpha} = \frac{p_1 - \sin^2 \alpha}{p_1 - 1} \quad \ldots\ldots\ldots\ldots\ldots \quad (63)$$

or

$$\sin^2 a = \frac{p - \sin^2 \alpha}{p\,(1 - \sin^2 \alpha)}$$

we obtain

$$S_s = \sqrt{\frac{(1 - p_1)\,(p_1 - \sin^2 \alpha)}{p_1}}\, \Pi\,(p_1, \alpha, \varphi) - \frac{\sin^2 \alpha}{\sqrt{-p \cdot p_1}}\, F\,(\alpha, \varphi)$$

According to LEGENDRE: "Traité des fonctions elliptiques", Vol. 1, page 72 (Ref. 1) this expression, with the above relation between p and p_1, is identical to

$$S_s = \sqrt{\frac{(1 - p)\,(p - \sin^2 \alpha)}{p}}\, \Pi\,(p, \alpha, \varphi) + \arctan \frac{\tan a \, \sin^2 \alpha \, \sin \varphi \, \cos \varphi}{\sqrt{1 - \sin^2 a \, \cos^2 \alpha} \cdot \sqrt{1 - \sin^2 \alpha \, \sin^2 \varphi}} \quad (64)$$

(which may easily be verified by differentiating both expressions). This expression may therefore replace (62).

As no tables are available giving the incomplete elliptic integrals of the third kind, it is generally necessary to evaluate the arc length from the differential expression (60), using Simpson's or some similar formula. If, however, only the arc length corresponding to $\varphi = 90°$ is to be computed, i.e. the arc length for a quadrant of the curve, this can be derived exactly.

In: "Tables of complete elliptic integrals", pp. 131—134 (Ref. 11) HEUMAN has introduced the integral

$$\left.\begin{aligned} \Lambda\,(\alpha, \beta, \varphi) &= \sqrt{\frac{(1 - p)\,(p - \sin^2 \alpha)}{p}}\, \Pi\,(p, \alpha, \varphi) \\ \text{where} \qquad p &= \frac{\sin^2 \alpha}{1 - \sin^2 \beta \, \cos^2 \alpha} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (65)$$

and the corresponding complete integrals

$$\Lambda\,(\alpha,\,\beta) = \frac{\pi}{2}\Lambda_0\,(\alpha,\,\beta)$$

By comparing the last equation in (65) with eq. (63), we find that $\beta = a$.

According to LEGENDRE, a complete elliptic integral of the third kind, and consequently also a complete Λ-integral, can, however, be expressed by elliptic integrals of the first and second kind. HEUMAN has given in his above-mentioned work [page 133, eq. (2)] in a somewhat modified form the relation

$$\left.\begin{aligned}\sqrt{\frac{(1-p)\,(p-\sin^2\alpha)}{p}}\;\Pi\,(p,\,\alpha) = \Lambda\,(\alpha,\,\beta) = F\,(\alpha)\cdot E\,(90^\circ-\alpha,\,\beta) - \\ -[F(\alpha) - E\,(\alpha)]\cdot F\,(90^\circ-\alpha,\,\beta)\end{aligned}\right\} \quad .. \quad (66)$$

where $\Pi\,(p,\,\alpha)$ is the complete elliptic integral of the third kind.

In HEUMAN: "Elliptiska integraler i dynamiska problem" (Elliptic integrals in dynamic problems) (Ref. 6) the above relation is derived on pages 216—217.

Since $\beta = a$, the arc length for a quadrant of the spherical ellipse, measured in radians, can now be expressed by the equation

$$S_{e\,1/4} = F\,(\alpha)\cdot E\,(90^\circ-\alpha,\,a) - [F\,(\alpha) - E\,(\alpha)]\cdot F\,(90^\circ-\alpha,\,a) = \Lambda\,(\alpha,\,a) = \frac{\pi}{2}\Lambda_0\,(\alpha,\,a) \quad (67)$$

In HEUMAN (Ref. 11) the function $\Lambda_0\,(\alpha,\,\beta)$ is tabulated to six decimal places. As can be seen from eq. (67), this function gives the arc length of a quadrant of the spherical ellipse measured in right angles. HEUMAN has used his Λ-integral in connection with the theory of the spherical and the gyroscopic pendulum. He seems not to have observed, however, that his complete Λ-integral simply gives the arc length for a quadrant of the spherical ellipse.

If we consider a cone with the apex in the mid-point of the sphere and with its base formed by the spherical ellipse, S_e is obviously also equal to the apex angle of the developed surface of the cone.

If the real length of the spherical curve is required, the arc length S, in radians, must be multiplied by the radius of the sphere.

If a table giving S_e as a function of φ is computed with the aid of Simpson's formula using the differential expression (60), it is possible to check very effectively by means of eq. (67) or HEUMAN's tables the accuracy of the computations.

As an example, a table giving S_e as a function of φ has been computed for the values

$$\alpha = 30^\circ \text{ and } a = 29.670^\circ$$

For the computations, a "Facit" calculating machine has been used having automatic multiplication and division, and reversible direction of rotation for the multiplier register. Since, when multiplying, the multiplier is the same for all φ-values, the computations are carried out very quickly. A table giving $\sin^2\varphi$ for every integer degree is used for the com-

putations. At first the expression $1 + \tan^2 a \sin^2 \alpha \sin^2 \varphi$ has been computed for all φ-values. This could be done in a single operation. Only the multiplier $\sin^2 \varphi$ had to be changed for the different φ-values. Next the expression $1 - \sin^2 \alpha \sin^2 \varphi$ has been computed in a similar way in a single operation, using reversed direction of rotation for the multiplier register. The square root has then been found from BARLOW's tables (Ref. 10) by means of interpolation in the calculating machine. Further the quotient $\frac{\sqrt{1 - \sin^2 \alpha \sin^2 \varphi}}{1 + \tan^2 a \sin^2 \alpha \sin^2 \varphi}$ has been computed using automatic division. Using Simpson's formula we now compute

$$3 \int_0^{\varphi} \frac{\sqrt{1 - \sin^2 \alpha \sin^2 \varphi}}{1 + \tan^2 a \sin^2 \alpha \sin \varphi} \, d\varphi$$

for all integer φ-values. Two series of computations must be carried out. For the first series we start from $\varphi = 0$, and obtain the values for all even numbers of degrees. For the second series we start from $\varphi = 1°$ and obtain the values for all odd numbers of degrees. Here, however, the value of S_e for $\varphi = 1°$ must first be determined, and this is also done by using Simpson's formula, observing that the ordinate for $\varphi = -1°$ is equal to the ordinate for $\varphi = +1°$. For each such series no primary values need to be fed more than once into the setting register, and the product register is never emptied. Finally the values thus computed have been multiplied by $\frac{\pi}{180} \cdot \frac{1}{3} \tan a \sqrt{1 - \sin^2 a \cos^2 \alpha}$ to obtain S. The values obtained are tabulated in Appendix 11, in which the differences ΔS have also been tabulated.

As a check of the accuracy of the computations, $S_{e^1/_4}$, i.e. S for $\varphi = 90°$, has also been computed from eq. (67) giving

$$S_{e^1/_4} = 1.6857504 \cdot 0.5008945 - [1.6857504 - 1.4674622]\ 0.5358467 = 0.727414$$

The value computed in the table is 0.72741, from which the conclusion may be drawn that the values in the table are so accurate that all five decimal places may be used.

If $S_{e^1/_4}$ is taken from HEUMAN's tables of $\Lambda_0\ (\alpha, \beta)$ using second degree interpolation, we obtain

$$S_{e^1/_4} = \frac{\pi}{2} \cdot \Lambda_0\ (30°,\ 29.670°) = \frac{\pi}{2} \cdot 0.463086 = 0.727414, \text{ i. e. the same value as that obtained above.}$$

B. The imaginary spherical ellipse

a) The equation of the imaginary spherical ellipse in great circle coordinates

Similarly as for the plane conic sections, a may be chosen imaginary in eq. (43), which is valid for the conic section with real axes. Also b then becomes imaginary if $e < 1$ and real. Eq. (43) then takes the form

$$\left.\begin{aligned} &\frac{\tan^2 x}{\tanh^2 |a|} + \frac{\tan^2 y}{\tanh^2 |b|} = -1 \\ &\cosh |b| = \frac{\cosh |a|}{\cosh e\,|a|} \end{aligned}\right\} \quad \text{where} \quad \left\{\begin{aligned} |a| &= \frac{a}{i} \\ |b| &= \frac{b}{i} \end{aligned}\right. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (68)$$

where i = the unit of imaginaries

We must have $e < 1$, since otherwise we obtain $\cosh |b| < 1$. Thus $\tanh^2 |b| < 0$, but this corresponds to a real spherical conic section. We shall call this curve the *imaginary spherical ellipse.*

We now introduce the substitutions [cf. eq. (44)]

$$\left.\begin{aligned} &\frac{\tan x}{\tan a} = \sin i\,u = \frac{1}{i}\tan\varphi \text{ or } \frac{\tan x}{\tanh |a|} = \tan\varphi \\ &\cosh u = \frac{1}{\cos\varphi} \text{ or } \tanh\frac{u}{2} = -\tan\frac{\varphi}{2} \end{aligned}\right\} \quad \text{from which} \quad \left\{ du = -\frac{1}{\cos\varphi}\,d\varphi \right. \qquad \ldots\ldots\ldots\ldots\ldots \quad (69)$$

If φ is real, x will also be real but y purely imaginary.

Using the substitutions (69), the eqs. (68) may be written in the parametric form

$$\left.\begin{aligned} \tan x &= \frac{1}{i}\tan a\ \tan\varphi = \tanh |a|\ \tan\varphi \\ \tan y &= \tan b\,\frac{1}{\cos\varphi} = i\tanh |b|\,\frac{1}{\cos\varphi} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots \quad (70)$$

b) The equation of the imaginary spherical ellipse in spherical polar coordinates

By comparing the substitutions according to eq. (44) for the real spherical conic section with the substitutions according to eqs. (68) and (69) for the imaginary spherical ellipse, it can be seen that we obtain the imaginary spherical ellipse, if in the equation of the real spherical conic section

$$\left.\begin{aligned} a &\quad \text{is replaced by} \quad i\,|a| \\ b &\quad \text{"} \quad \text{"} \quad \text{"} \quad i\,|b| \\ \sin\varphi &\quad \text{"} \quad \text{"} \quad \text{"} \quad \frac{1}{i}\tan\varphi \\ \cos\varphi &\quad \text{"} \quad \text{"} \quad \text{"} \quad \frac{1}{\cos\varphi} \\ d\varphi &\quad \text{"} \quad \text{"} \quad \text{"} \quad i\,du = \frac{1}{i\cos\varphi}\,d\varphi \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots \quad (71)$$

Thus the equation in polar coordinates may be obtained by making these substitutions in eqs. (52) and (53), and we then have the equation for the imaginary spherical ellipse in polar coordinates, in parametric form

$$\left.\begin{aligned} &\tan\,(r - i\,|a|) = \tanh\,|a|\,\sin\alpha\,\tan\varphi \\ &\tan\frac{\vartheta}{2} = \tan\frac{90^\circ - \alpha}{2}\,(\cos\varphi + i\,\sin\varphi) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (72)$$

where

$$\left.\begin{aligned} &\sin\alpha = \frac{\tanh\,e\,|a|}{\tanh\,|a|} \\ &e \leq 1 \end{aligned}\right\}$$

For the derivation of the second equation in (72), $\tan\frac{90^\circ - \varphi}{2}$ in eqs. (52) is first rewritten as $\frac{1 - \sin\varphi}{\cos\varphi}$, which, after substitution from (71), becomes $\cos\varphi + i\,\sin\varphi$.

The equation in polar coordinates in non-parametric form becomes

$$\tan r = \frac{i\,\tanh\,|a|\,\cos^2\alpha}{1 - \tanh^2|a|\,\sin^2\alpha - \sin\alpha\,(1 - \tanh^2|a|)\,\cos\vartheta} \quad \ldots\ldots\ldots \quad (73)$$

As the main curve, we define the curve obtained if $r - i\,|a|$ according to eq. (72) is chosen in the same quadrant as φ. As the antipode curve, we define the curve obtained if $r - i\,|a|$ lies in the same quadrant as $\varphi + 180^\circ$.

c) The angle between the radius vector and the normal

The angle from the focal radius to the normal to the curve may also be obtained by making the above substitutions (71) in eq. (58) for the real spherical conic section. This gives

$$\tan\Phi = \pm\frac{\tan\alpha}{\cosh\,|a|\,\sqrt{1 - (1 - \tanh^2|a|\,\sin^2\alpha)\,\sin^2\varphi}} \quad \ldots\ldots\ldots \quad (74)$$

It will now be shown that in eq. (74)
the positive sign corresponds to the main curve and
" negative " " " " antipode curve.

According to the first equation in (72), $\frac{dr}{d\varphi}$ is always positive. According to eq. (54), therefore, $\tan\Phi$ becomes positive if $\frac{d\vartheta}{d\varphi}\sin r$ is negative and vice versa. Both $\frac{d\vartheta}{d\varphi}$ and $\sin r$ are complex, but the product must be real because $\tan\Phi$ according to eq. (74) is real. To have the product of two complex numbers real, one number must equal a real constant times the conjugate value of the other number. If the product shall be positive, we must therefore have the product of the real parts of both numbers positive, and vice versa.

If the above substitutions (71) are inserted in (56), we obtain

$$\frac{d\vartheta}{d\varphi} = -\frac{\cos\alpha}{\sin\alpha\,\sin\varphi + i\,\cos\varphi} = -\frac{\cos\alpha\,\sin\alpha\,\sin\varphi + i\,\cos\alpha\,\cos\varphi}{1 - \cos^2\alpha\,\sin^2\varphi}$$

The real part in $\frac{d\vartheta}{d\varphi}$ therefore becomes negative if $0 < \varphi < 180°$, and positive if $180° < \varphi < 360°$.

From the first equation in (72) it can be seen that the real part of r is equal to $r - i\,|a|$. We therefore have $\sin r = \cosh |a| \sin (r - i\,|a|) + i \sinh |a| \cos (r - i\,|a|)$. For the main curve where $r - i\,|a|$ and φ lie in the same quadrant the real part of $\sin r$ is therefore positive if $0 < \varphi < 180°$, and negative if $180° < \varphi < 360°$. For the main curve, therefore, $\frac{d\vartheta}{d\varphi} \sin r$ is always negative, i.e. $\tan \Phi$ is positive, which corresponds to the positive sign in eq. (74). For the antipode curve where $r - i\,|a| + 180°$ and φ lie in the same quadrant, the above product becomes positive, i.e. $\tan \Phi$ is negative, which corresponds to the negative sign in eq. (74).

Using the first equation in (72), eq. (74) may be rewritten as

$$\tan \Phi = \frac{\tan \alpha \cos (r - i\,|a|)}{\cosh |a| \cos \varphi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (75)$$

In this last equation no double signs are required, since $\frac{\cos (r - i\,|a|)}{\cos \varphi}$ is always positive for the main curve, and negative for the antipode curve.

d) The arc length of the imaginary spherical ellipse

By making the substitutions according to eqs. (71) in eq. (60) valid for the real spherical conic section, we obtain for the arc length S_{ei} of the imaginary spherical ellipse

$$dS_{ei} = \sinh |a| \sqrt{1 - \tanh^2 |a| \sin^2 \alpha}\, \frac{\sqrt{1 - \cos^2 \alpha \sin^2 \varphi}}{1 - (1 - \tanh^2 |a| \sin^2 \alpha) \sin^2 \varphi}\, d\varphi \quad .. \quad (76)$$

We introduce the substitutions

$$\left.\begin{aligned} \alpha' &= 90° - \alpha \\ \Delta'\varphi &= \sqrt{1 - \sin^2 \alpha' \sin^2 \varphi} \\ p &= 1 - \tanh^2 |a| \sin^2 \alpha \text{ where } p \text{ is positive} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (77)$$

from which

$$\tanh^2 |a| = \frac{1 - p}{1 - \sin^2 \alpha'}$$

$$\sinh^2 |a| = \frac{1 - p}{p - \sin^2 \alpha'}$$

Eq. (76) then takes the form

$$dS_{ei} = \frac{\sinh |a| \sqrt{1 - \tanh^2 |a| \sin^2 \alpha}}{1 - p \sin^2 \varphi} \cdot \frac{1 - \sin^2 \alpha' \sin^2 \varphi}{\Delta'\varphi}\, d\varphi$$

or

$$dS_{ei} = \sinh |a| \sqrt{1 - \tanh^2 |a| \sin^2 \alpha} \left[\frac{1 - \frac{\sin^2 \alpha'}{p}}{(1 - p \sin^2 \varphi)\, \Delta'\varphi} + \frac{\frac{\sin^2 \alpha'}{p}}{\Delta'\varphi} \right] d\varphi$$

or, if $\mathfrak{p}$ from eqs. (77) is substituted in the numerators,

$$dS_{ei} = \frac{\tanh |a|}{\cosh |a|} \cdot \frac{\sin^2 \alpha}{\sqrt{1 - \tanh^2 |a| \sin^2 \alpha}} \cdot \frac{d\varphi}{(1 - \mathfrak{p} \sin^2 \varphi)\, \Delta' \varphi} + \frac{\sinh |a| \cos^2 \alpha \, d\varphi}{\Delta' \varphi \sqrt{1 - \tanh^2 |a| \sin^2 \alpha}}$$

By integrating this expression, we obtain, if S_{ei} is measured from the point where $\varphi = 0$,

$$S_{ei} = \frac{\tanh |a|}{\cosh |a|} \cdot \frac{\sin^2 \alpha}{\sqrt{1 - \tanh^2 |a| \sin^2 \alpha}} \Pi (\mathfrak{p}, \alpha', \varphi) + \frac{\sinh |a| \cos^2 \alpha}{\sqrt{1 - \tanh^2 |a| \sin^2 \alpha}} F (\alpha', \varphi) \qquad (78)$$

which can be transformed into

$$S_{ei} = \sqrt{\frac{(1 - \mathfrak{p})\ (\mathfrak{p} - \sin^2 \alpha')}{\mathfrak{p}}}\, \Pi (\mathfrak{p}, \alpha', \varphi) + \sqrt{\frac{1 - \mathfrak{p}}{\mathfrak{p}\ (\mathfrak{p} - \sin^2 \alpha')}}\, \sin^2 \alpha' \cdot F (\alpha', \varphi) \qquad (79)$$

The first term is an elliptic integral of the third kind. Since $1 > \mathfrak{p} > \sin^2 \alpha'$, it belongs to the circular class. According to eq. (65) the expression can, however, be rewritten as

$$S_{ei} = \Lambda (\alpha', \beta, \varphi) + \sqrt{\frac{(1 - \mathfrak{p})}{\mathfrak{p}\ (\mathfrak{p} - \sin^2 \alpha')}}\, \sin^2 \alpha' \cdot F (\alpha', \varphi) \quad \ldots\ldots\ldots \qquad (80)$$

where the relation between $\mathfrak{p}$ and β is given by

$$\mathfrak{p} = \frac{\sin^2 \alpha'}{1 - \cos^2 \alpha' \sin^2 \beta}$$

Substituting $\mathfrak{p}$ from eqs. (77) and solving with respect to β we obtain

$$\tan \beta = \frac{1}{\sinh |a| \cos \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (81)$$

As no tables are available giving the elliptic integrals of the third kind, it is generally necessary to compute the arc length from the differential expression (76), using Simpson's formula. For the complete integral, however, we obtain according to eq. (66)

$$\left.\begin{aligned} S_{ei\,1/4} &= F (\alpha') \cdot E (\alpha, \beta) - [F (\alpha') - E (\alpha')] \cdot F (\alpha, \beta) + \frac{\sinh |a| \cos^2 \alpha}{\sqrt{1 - \tanh^2 |a| \sin^2 \alpha}} F (\alpha') \\ &\text{where} \\ &\alpha' = 90^\circ - \alpha \text{ and } \tan \beta = \frac{1}{\sinh |a| \cos \alpha} \end{aligned}\right\} \qquad (82)$$

which gives the arc length of a quadrant of the imaginary spherical ellipse.

C. The imaginary spherical hyperbola

a) The equation of the imaginary spherical hyperbola in great circle coordinates

If in eqs. (43), which apply to the real spherical conic section, we make both a and e imaginary, the equations take the form

$$\left.\begin{aligned} &\frac{\tan^2 x}{\tanh^2 |a|} + \frac{\tan^2 y}{\tanh^2 |b|} = -1 \\ &\text{where} \\ &\cosh |b| = \frac{\cosh |a|}{\cos |e|\,|a|} \\ &|a| = \frac{a}{i} \\ &|b| = \frac{b}{i} \\ &|e| = \frac{e}{i} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (83)$$

The curve according to the above equations will be called here the *imaginary spherical hyperbola*. Eqs. (83) are in form the same equations as eqs. (68) for the imaginary spherical ellipse, except that, whereas for (68) we have $|b| < |a|$, here we have $|b| > |a|$.

For the imaginary spherical ellipse, x was assumed to be real and then y became imaginary. For the spherical imaginary hyperbola we make y real, and x imaginary by means of the substitutions

$$\left.\begin{aligned} &\frac{\tan ea}{\tan a} = \sin \bar{\alpha} = \frac{i \tan |e|\,|a|}{\tanh |a|} = i \sinh |\bar{\alpha}| = \frac{i}{\tan \alpha} \\ &\sqrt{1 + \frac{\tan^2 (|e|\,|a|)}{\tanh^2 |a|}} = \frac{1}{\sin \alpha} \\ &\frac{\tan x}{\tan a} = \sin (90^\circ - iu) = \cosh u = \frac{1}{\sin \varphi} \text{ or } \frac{\tan x}{\tanh |a|} = \frac{i}{\sin \varphi} \\ &\cos (90^\circ - iu) = i \sinh u = \frac{i}{\tan \varphi} \\ &\text{from which} \\ &d (90^\circ - iu) = \frac{i}{\sin \varphi}\, d\varphi \end{aligned}\right\} \quad \ldots\ldots \quad (84)$$

If (84) is substituted in eqs. (83), the equation for the imaginary spherical hyperbola may now be written as

$$\left.\begin{aligned} \tan x &= i \tanh |a| \frac{1}{\sin \varphi} \\ \tan y &= \tanh |b| \frac{1}{\tan \varphi} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (85)$$

where x is purely imaginary and y is real.

The imaginary spherical hyperbola, as may be seen from the above, is the same type of curve as the imaginary spherical ellipse, but with the axes x and iy replaced by y and ix. The foci, however, are located differently. For the imaginary spherical ellipse the distance from the origin to the focus is equal to $ea = i\, e|a|$, and is therefore imaginary. For the imaginary spherical hyperbola it is equal to $ea = |e|\,|a|$ and is real. For the imaginary spherical hyperbola the eccentricity e can assume any value between 0 and $+ i\,\infty$.

b) The equation of the imaginary spherical hyperbola in polar coordinates

By comparing the substitutions according to eqs. (43), (44) and (46) for the real spherical conic section with the substitutions according to equations (83) and (84) for the imaginary spherical hyperbola, we find that we can obtain the equation of the imaginary spherical hyperbola, if in the equation of the real conic section

$$\left.\begin{array}{lll} a & \text{is replaced by} & i\,|a| \\ b & \text{"} \quad \text{"} \quad \text{"} & i\,|b| \\ e & \text{"} \quad \text{"} \quad \text{"} & i\,|e| \\ \sin\alpha & \text{"} \quad \text{"} \quad \text{"} & \dfrac{i}{\tan\alpha} \\ \cos\alpha & \text{"} \quad \text{"} \quad \text{"} & \dfrac{1}{\sin\alpha} \\ \sin\varphi & \text{"} \quad \text{"} \quad \text{"} & \dfrac{1}{\sin\varphi} \\ \cos\varphi & \text{"} \quad \text{"} \quad \text{"} & \dfrac{i}{\tan\varphi} \\ d\varphi & \text{"} \quad \text{"} \quad \text{"} & d\,(90^\circ - iu) = \dfrac{i}{\sin\varphi}\,d\varphi \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (86)$$

It is therefore possible to obtain the equation for the imaginary spherical hyperbola in polar coordinates in parametric form by making the substitutions (86) in eqs. (52). If these substitutions are made, we obtain

$$\left.\begin{array}{l} \tan\,(r - i\,|a|) = -\tanh|a|\,\dfrac{1}{\tan\alpha}\cdot\dfrac{1}{\sin\varphi} \\ \tan\dfrac{\vartheta}{2} = (\cos\alpha + i\,\sin\alpha)\,\tan\dfrac{90^\circ - \varphi}{2} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (87)$$

where

$$\left.\tan\alpha = \frac{\tanh|a|}{\tan\,(|e|\,|a|)}\right\}$$

The equation in polar coordinates in non-parametric form is obtained if the same substitutions are made in eqs. (53)

$$\tan r = \frac{i\,\tanh|a|}{\sin^2\alpha + \tanh^2|a|\,\cos^2\alpha - i\,\sin\alpha\,\cos\alpha\,(1 - \tanh^2|a|)\,\cos\vartheta} \quad \ldots \quad (88)$$

c) The angle from the radius vector to the normal

This angle also may be obtained from the corresponding equation (58) by making the substitutions according to (86). This gives after simplification

$$\tan\Phi = \pm\,\frac{\sin\alpha\,\cos\varphi}{\sinh|a|\,\sqrt{1 + \dfrac{\tan^2\alpha}{\tanh^2|a|}\,\sin^2\varphi}} \quad \ldots\ldots\ldots\ldots\ldots \quad (89)$$

As the main curve, we define the curve obtained if $r - i\,|a|$ is chosen in the first or second quadrant, i.e. $[r - (90° + i\,|a|)]$ in the first or fourth quadrant. As the antipode curve, we define the curve obtained if $[r - (90° + i\,|a|)]$ lies in the second or third quadrant. It will now be shown that in eq. (89)

the positive sign corresponds to the main curve and
" negative " " " " antipode curve.

By differentiating the first equation in (87), we find that $\frac{dr}{d\varphi} \cdot \frac{1}{\cos\varphi}$ is always real and positive. In eq. (54), therefore, $\tan \Phi$ becomes positive if $\frac{d\vartheta}{d\varphi} \cdot \frac{\sin r}{\cos\varphi}$ is negative, and vice versa. Both $\frac{d\vartheta}{d\varphi}$ and $\sin r$ are complex, but the product must be real because $\tan \Phi$ must be real To get the product of two complex quantities real and positive, we must, as explained on page 31, have the product of the real parts in both quantities positive and vice versa.

If the substitutions according to (86) are made in (56), we obtain for the imaginary spherical hyperbola

$$\frac{d\vartheta}{d\varphi} = \frac{-i}{\sin\varphi \sin\alpha \left(1 + \frac{i}{\tan\alpha} \cdot \frac{1}{\sin\varphi}\right)} = \frac{-i}{\sin\alpha \left(\sin\varphi + \frac{i}{\tan\alpha}\right)} = \frac{-\frac{1}{\tan\alpha} - i\sin\varphi}{\sin\alpha \left(\sin^2\varphi + \frac{1}{\tan^2\alpha}\right)}$$

The real part in $\frac{d\vartheta}{d\varphi}$ is therefore always negative. From the first equation in (87) it can be seen that the real part of r equals $r - i\,|a|$. We have

$$\sin r = \cos\,[r - (i\,|a| + 90°) + i\,|a|] = \cosh|a| \cos\,[r - (90° + i\,|a|)] - \\ - i \sinh|a| \cdot \sin\,[r - (90° + i\,|a|)].$$

The real part of $\sin r$ is consequently always positive for the main curve, and negative for the antipode curve. For the main curve the product $\frac{d\vartheta}{d\varphi} \sin r$ thus becomes negative, and $\tan \Phi$ has the same sign as $\cos\varphi$, corresponding to the upper sign in eq. (89). For the antipode curve, $\tan \Phi$ instead has the opposite sign to $\cos\varphi$ and corresponds to the lower sign in eq. (89).

By using the first equation in (87), eq. (89) may be rewritten in the form

$$\tan \Phi = \frac{\sin\alpha \cos\,[r - (90° + i\,|a|)] \cos\varphi}{\sinh|a|} \qquad \ldots\ldots\ldots\ldots\ldots \quad (90)$$

where the $\pm$ signs are not required.

d) The arc length of the imaginary spherical hyperbola

Eq. (60) is the general differential expression for the arc length of the spherical conic section. If we make the substitutions (86) in (60), we obtain the following differential expression for the arc length S_{hi} of the imaginary spherical hyperbola

$$dS_{hi} = -\frac{\sinh |a|}{\sin\alpha \sqrt{1 + \frac{\tanh^2 |a|}{\tan^2 \alpha}}} \cdot \frac{\sqrt{1 - \sin^2 \alpha \cos^2 \varphi}}{1 - \frac{1}{1 + \frac{\tanh^2 |a|}{\tan^2 \alpha}} \cos^2 \varphi} d\varphi \quad \ldots\ldots\ldots \quad (91)$$

This expression leads to an elliptic integral of the third kind. It is therefore generally necessary to compute the arc lengths from this expression using Simpson's formula.

If in eq. (76)

$$\left.\begin{array}{lll} \varphi & \text{is replaced by} & 90^\circ - \varphi \\ \alpha & \text{" " "} & 90^\circ - \alpha \\ \sinh |a| & \text{" " "} & \dfrac{\sinh |a|}{\sin \alpha} \\ S_e & \text{" " "} & S_{hi} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (92)$$

thus also

$$\tanh^2 |a| \text{ is replaced by } \frac{\sinh^2 |a|}{\sin^2 \alpha + \sinh^2 |a|}$$

then eq. (76) takes the form

$$dS_{hi} = -\frac{\sinh |a|}{\sin \alpha} \sqrt{1 - \frac{\sinh^2 |a| \cos^2 \alpha}{\sin^2 \alpha + \sinh^2 |a|}} \cdot \frac{\sqrt{1 - \sin^2 \alpha \cos^2 \varphi}}{1 - \left(1 - \frac{\sinh^2 |a| \cos^2 \alpha}{\sin^2 \alpha + \sinh^2 |a|}\right) \cos^2 \varphi} d\varphi$$

and this expression may be transformed into

$$dS_{hi} = -\frac{\sinh |a|}{\sin\alpha \sqrt{1 + \frac{\tanh^2 |a|}{\tan^2 \alpha}}} \cdot \frac{\sqrt{1 - \sin^2 \alpha \cos^2 \varphi}}{1 - \frac{1}{1 + \frac{\tanh^2 |a|}{\tan^2 \alpha}} \cos^2 \varphi} d\varphi$$

which is identical with eq. (91).

If we have an imaginary spherical hyperbola with the constants $|a|$ and α, its arc length may thus be obtained from the expression for the arc length for an imaginary spherical ellipse, provided that we make the substitutions (92) in the expression for the arc length of the last-mentioned curve.

The integral of (91) from 90° to φ can therefore be obtained by making the substitutions (92) in the integrals (78), (79) or (82).

After simplification, we obtain

$$\left.\begin{array}{l} S_{hi} = \dfrac{\tanh |a| \cos^2 \alpha}{\sqrt{\sin^2 \alpha + \sinh^2 |a|}} \, \Pi\,(p, \alpha, \varphi') + \tanh |a| \sqrt{\sin^2 \alpha + \sinh^2 |a|} \, F\,(\alpha, \varphi') \\ \text{where} \\ p = 1 - \dfrac{\cos^2 \alpha \sinh^2 |a|}{\sin^2 \alpha + \sinh^2 |a|} = \dfrac{1}{1 + \dfrac{\tanh^2 |a|}{\tan^2 \alpha}} \\ \varphi' = 90^\circ - \varphi \\ \text{and } S_{hi} \text{ is measured from the point where } \varphi = 90^\circ \end{array}\right\} \quad (93)$$

or, according to eq. (79),

$$S_{hi} = \sqrt{\frac{(1-p)\,(p-\sin^2\alpha)}{p}}\;\Pi\;(p,\;\alpha,\;\varphi') + \sqrt{\frac{1-p}{p\,(p-\sin^2\alpha)}}\;\sin^2\alpha\cdot F\;(\alpha,\;\varphi') \quad (94)$$

and the arc length for a quadrant from $\varphi = 90°$ to $\varphi = 0°$ according to eq. (82)

$$\left.\begin{aligned} S_{hi\,1/4} &= F\;(\alpha)\cdot E\;(\alpha',\;\beta) - [F\;(\alpha) - E\;(\alpha)]\cdot F\;(\alpha',\;\beta) + \\ &\quad + \tanh|a|\,\sqrt{\sin^2\alpha + \sinh^2|a|}\cdot F\;(\alpha) \end{aligned}\right\} \quad \ldots\ldots\ldots (95)$$

where

$$\left.\begin{aligned} &\alpha' = 90° - \alpha \text{ and } \tan\beta = \frac{1}{\sinh|a|} \text{ or } \beta = 90° - \operatorname{amh}|a| \end{aligned}\right\}$$

S_{hi} decreases with increasing values of φ and is measured from the point where $\varphi = 90°$.

D. The limiting curve to the imaginary spherical conic section curves when the eccentricity $e \to 0$

a) The equation of the curve in polar coordinates

If in eq. (70) $\varphi \to \pm\,90°$ and $e \to 0$, thus according to eq. (72) also $\alpha \to 0$ and according to eq. (68) $|b| \to |a|$, then according to eq. (70) both x and y will be infinite and, from the parametric eqs. (71), r will be indeterminate and $\vartheta \to \pm\, i\,\infty$. The curve nevertheless approaches a definite limiting curve, as will be shown below. If in eq. (73) for the imaginary spherical ellipse we let $\alpha \to 0$, the equation becomes

$$\tanh r = \frac{i\,\tanh|a|}{1 - \sin\alpha\,(1 - \tanh^2|a|)\cos\vartheta}$$

We now rotate the polar axis through the angle $\pm\, iw$, where $w \to +\,\infty$, by putting $\overline{\vartheta} = \vartheta \mp iw$ or $\vartheta = \overline{\vartheta} \pm iw$

where the upper sign is valid if $\varphi \to +\,90°$
" lower " " " " " $\varphi \to -\,90°$

The preceding equation for the curve then becomes

$$\tanh r = \frac{i\,\tanh|a|}{1 - \sin\alpha\,(1 - \tanh^2|a|)\cos(\overline{\vartheta} \pm iw)}$$

Expanding $\cos(\overline{\vartheta} \pm iw)$, we have

$\cos(\overline{\vartheta} \pm iw) = \cos\overline{\vartheta}\cosh w \mp i\sin\overline{\vartheta}\sinh w = \cosh w\,[\cos\overline{\vartheta} \mp i\tanh w\sin\overline{\vartheta}]$. Since $w \to \infty$ then $\tanh w \to 1$, and the equation of the curve may be written

$$\tan r = \frac{i\,\tanh|a|}{1 - \sin\alpha\,(1 - \tanh^2|a|)\,(\cosh w)\,e^{\mp\, i\overline{\vartheta}}}$$

where e = the base of the natural system of logarithms (when used in a power expression; otherwise e denotes the eccentricity).

If we now let $w \to +\infty$ and $\alpha \to 0$ in such a way that $\sin \alpha\,(1 - \tanh^2|a|) \cosh w \to 1$, and if we omit the bar in $\bar{\vartheta}$, the expression becomes

$$\tan r = \frac{i \tanh |a|}{1 - e^{\mp i\vartheta}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (96)$$

which is the equation of the curve. It can be seen that this curve consists of two sets of branches which are the mirror images of each other about the polar axis.

One set of branches

$$\tan r = \frac{i \tanh |a|}{1 - e^{-i\vartheta}}$$

corresponds to $\varphi \to +90°$ for the imaginary spherical ellipse, and is denoted by A. The other set of branches

$$\tan r = \frac{i \tanh |a|}{1 - e^{+i\vartheta}}$$

corresponds to $\varphi \to -90°$ for the imaginary spherical ellipse, and is denoted by B.

Instead of beginning from the imaginary spherical ellipse, it is also possible to begin from the imaginary spherical hyperbola, if in its eq. (88) we let $e \to 0$, i.e. according to eqs. (87) $\alpha \to 90°$. Then the equation becomes

$$\tan r = \frac{i \tanh |a|}{1 - i \cos \alpha\,(1 - \tanh^2 |a|) \cos \vartheta}$$

Here too we rotate the polar axis by making $\vartheta = \bar{\vartheta} \pm iw - 90°$. The above equations can then be written in the form

$$\tan r = \frac{i \tanh |a|}{1 - \cos \alpha\,(1 - \tanh^2 |a|)\,(\sinh w)\, e^{\mp i\bar{\vartheta}}}$$

If now $w \to \infty$ and $\alpha \to 90°$ in such a way that $\cos \alpha\,(1 - \tanh^2|a|) \sinh w \to 1$, and if we omit the bar in $\bar{\vartheta}$, we again obtain eq. (96), which is thus the limiting case for both the imaginary spherical ellipse and the imaginary spherical hyperbola when $e \to 0$.

Eq. (96) may be transformed into parametric form in the following manner. We introduce the substitution

$$\tan (r - i\,|a|) = \tanh |a| \tan \varphi$$

from which

$$\frac{\tan r - i \tanh |a|}{1 + i \tanh |a| \tan r} = \tanh |a| \tan \varphi$$

or, if this expression is solved with respect to $\tan r$,

$$\tan r = \frac{(i + \tan \varphi) \tanh |a|}{1 - i \tanh^2 |a| \tan \varphi}$$

From eq. (96) we have

$$e^{\mp i\vartheta} = 1 - \frac{i \tanh |a|}{\tan r}$$

If the value for tan r is substituted in this equation, we obtain

$$e^{\mp i\vartheta} = \frac{\tan \varphi}{(i + \tan \varphi) \cosh^2 |a|}$$

The equation in parametric form now becomes

$$\left.\begin{aligned} \tan\,(r - i\,|a|) &= \tanh |a|\ \tan \varphi \\ e^{\mp i\vartheta} &= \frac{\tan \varphi}{(\cosh^2 |a|)\,(i + \tan \varphi)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ (97)$$

or

$$\mp i\vartheta = \ln \frac{\tan \varphi}{(\cosh^2 |a|)\,(i + \tan \varphi)}$$

We obtain both the A- and B-branches even if φ only varies through 180°. It is, however, convenient to reserve the range between 0 and 180° for the A-branches corresponding to the — sign in front of $i\vartheta$, and the range 180° to 360° for the B-branches corresponding to the + sign in front of $i\vartheta$. The curve obtained if in eqs. (97) we choose $r - i\,|a|$ in the same quadrant as φ is considered as the main curve. The antipode curve is obtained if we choose $r - i\,|a|$ in the same quadrant as $\varphi + 180°$.

For these curves it can be seen from eqs. (97) that r is complex, but with the imaginary part constant and equal to $i\,|a|$. Any mating curve will therefore be a curve in the real spherical surface if the angle between the axes of rotation is complex with the imaginary part $= i\,|a|$.

b) The angle between the radius vector and the normal

The angle Φ from the radius vector to the normal to the curve is determined by eq. (54). Eq. (96) gives

$$\frac{dr}{\cos^2 r} = \frac{i\,(\mp i e^{\mp i\vartheta})\tanh |a|}{(1 - e^{\mp i\vartheta})^2}\,d\vartheta = \frac{\mp e^{\mp i\vartheta} \tan^2 r}{\tanh |a|}\,d\vartheta$$

from which

$$\frac{dr}{d\vartheta} = \mp \frac{e^{\mp i\vartheta} \sin^2 r}{\tanh |a|}$$

which, substituted in (54) and together with eq. (96) after simplification, gives

$$\tan \Phi = \pm \frac{\cos r}{\tanh |a|}\,(\tan r - i \tanh |a|) = \pm \frac{\sin\,(r - i\,|a|)}{\sinh |a|}$$

where the + sign applies to the A-branches
" — " " " " B "

For the A-branches, however, $\sin(r - i\,|a|)$ is always positive for the main curve and negative for the antipode curve. For the B-branches the reverse is the case. The above expression may therefore be written

$$\tan \Phi = \pm \frac{|\sin\,(r - i\,|a|)|}{\sinh |a|} \quad \ldots\ldots\ldots\ldots\ldots\ (98)$$

where the + sign applies for the main curve
" — " " " " " antipode curve

From the first equation in (97) we obtain

$$|\sin(r - i|a|)| = \frac{\tanh|a|\,|\tan\varphi|}{\sqrt{1+\tanh^2|a|\tan^2\varphi}}$$

which substituted in eq. (98) gives

$$\tan\Phi = \pm\frac{1}{\sqrt{\frac{\cosh^2|a|}{\sin^2\varphi}-1}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (99)$$

where the same rules for the signs apply as for eq. (98).

c) The arc length of the curve

For the arc length of a spherical curve we have the expression

$$dS = \sqrt{1+\sin^2 r\left(\frac{d\vartheta}{dr}\right)^2}\cdot\frac{dr}{d\varphi}d\varphi$$

Differentiating the first equation in (97) gives

$$\frac{dr}{\cos^2(r - i|a|)} = (1+\tanh^2|a|\tan^2\varphi)\,dr = \frac{\tanh|a|}{\cos^2\varphi}d\varphi$$

or

$$\frac{dr}{d\varphi} = \frac{\tanh|a|}{\cos^2\varphi+\tanh^2|a|\sin^2\varphi} = \frac{\sinh|a|\cosh|a|}{\sinh^2|a|+\cos^2\varphi}$$

Furthermore, we have according to eq. (54)

$$\sin^2 r\left(\frac{d\vartheta}{d\varphi}\right)^2 = \frac{1}{\tan^2\Phi}$$

or using eq. (99)

$$\sin^2 r\left(\frac{d\vartheta}{d\varphi}\right)^2 = \frac{\cosh^2|a|}{\sin^2\varphi}-1$$

If the above values of $\frac{dr}{d\varphi}$ and $\sin^2 r\left(\frac{d\vartheta}{d\varphi}\right)^2$ are substituted in the above expression for dS, and if the arc length of the curve is denoted by S_{exp}, we obtain for the arc length the expression

$$dS_{exp} = \frac{\sinh|a|\cosh^2|a|}{|\sin\varphi|\,(\sinh^2|a|+\cos^2\varphi)}d\varphi =$$

$$= \mp\left[\sinh|a|\,\frac{d(\cos\varphi)}{1-\cos^2\varphi} + \frac{d(\cos\varphi)}{\sinh|a|\left(1+\frac{\cos^2\varphi}{\sinh^2|a|}\right)}\right]$$

which, when integrated, where S_{exp} is measured from either of the points $\varphi = +90°$ or $\varphi = -90°$, gives the following expression

$$S_{exp} = \pm \left[\sinh |a| \ln \left| \tan \frac{\varphi}{2} \right| - \arctan \frac{\cos \varphi}{\sinh |a|} \right] \quad \ldots\ldots\ldots\ldots \quad (100)$$

where the + sign applies to the A-branches
" — " " " " B "

As opposed to the expression for the previous curves, the arc length for this curve is expressed in elementary functions.

In Chapter V : D a 2α and H it will be shown that the spherical exponential curve according to eq. (211) is a rolling curve to the type of curve treated here, and that the arc length for given values of φ will be the same for both curves. This is the explanation of the use of the symbol S_{exp} for the arc length.

CHAPTER V

ROLLING CURVES TO THE SPHERICAL CONIC SECTION CURVES

A. Rolling curves to the real spherical conic section

a) Deduction of the equations of the rolling curves

1) Rolling curves with real periodic number

In addition to the symbols already introduced, the following will also be used:

ν = periodic number of the rolling curve, i.e. the number of periods per 360° of polar angle. When used as a suffix, ν (or the corresponding number) indicates that the value refers to a ν-period curve

s = suffix indicating an initial or starting value

e = " " " end value

For the plane rolling curves, the condition for two curves to form a rolling curve pair is that the distance between the axes of rotation must equal the difference between the radii to the point of contact, and that the gear ratio must equal the ratio of the radii. For spherical rolling curves, the condition is instead expressed by eqs. (7). This condition means that the angle between the centres of rotation must equal the difference between the two radius vectors to the point of contact, and that the gear ratio must equal the ratio between the sines of the geodetic radius vectors. In eqs. (7) the sign for r_R must be selected in a definite manner, namely so that r_R has the same sign as r_L with internal meshing, but the opposite sign to r_L with external meshing and double internal meshing. Furthermore, we assume that $-180° < r < +180°$. For spherical rolling curves the nature of meshing may be defined in the following way. If no part of the radii coincides (the point of contact then lies between the centres of rotation), we have external meshing or, if $|C_{L-R}| > 180°$, double internal meshing. If the radii partly coincide (the point of contact then lies on the same side of the centres of rotation), we have internal meshing.

If we assume that $-180° < r < +180°$, an internal meshing can always be reduced to an external meshing by replacing the centre of rotation for the curve having the greater value of $|r|$ with its antipode. Furthermore, a double internal meshing, where the point of contact of the rolling curves lies on the greater arc of the great circle through the centres of rotation, may be reduced to a normal external meshing having the point of contact lying on the smaller arc of the great circle. This can be obtained by replacing both centres of rotation with the antipodes. Bevel gears may therefore always be treated as external gears. When deducing the equations of the rolling curves, however, we always consider the origin as the centre of rotation and distinguish between external and internal meshing.

By means of eqs. (7) we shall now form a mating curve to the real spherical conic section with the angle C_ν between the centres of rotation. If the real spherical conic section is regarded as the L-curve and the values belonging to it are given the suffix 1, since it is a single-period curve, and if the values belonging to the ν-period curve are given the suffix ν, we obtain

$$\frac{d\gamma_\nu}{d\gamma_1} = \frac{\sin r_1}{\sin r_\nu} = \frac{\sin r_1}{\sin (r_1 - C_\nu)}$$

We introduce the quantities $\varkappa_\nu$ and a_ν defined by

$$\left.\begin{aligned} a_\nu &= \varkappa_\nu\, a_1 \\ C_\nu &= a_1\ (1 \mp \varkappa_\nu) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (101)$$

where $\varkappa_\nu$ is positive.

If $C_\nu > a_1$, the $+$ sign applies, and if $C_\nu < a_1$, the $-$ sign applies. Since the angle of rotation γ_1 equals the polar angle ϑ_1, we obtain, if eqs. (101) are substituted in the preceding one,

$$\frac{d\gamma_\nu}{d\vartheta_1} = \frac{\sin r_1}{\sin [r_1 - a_1\ (1 \mp \varkappa_\nu)]}$$

which can be rewritten as

$$\frac{d\gamma_\nu}{d\vartheta_1} = \frac{\sin r_1}{\sin\ (r_1 - a_1)\ \cos \varkappa_\nu\, a_1 \pm \cos\ (r_1 - a_1)\ \sin \varkappa_\nu\, a_1}$$

We shall now substitute eqs. (52) for the spherical ellipse in this equation. The first equation in (52) gives, if the quantities r, a, α, ϑ and φ are given the suffix 1, since it is a single-period curve,

$$\sin\ (r_1 - a_1) = \pm \frac{\tan a_1 \sin \alpha_1 \sin \varphi_1}{\sqrt{1 + \tan^2 a_1 \sin^2 \alpha_1 \tan^2 \varphi_1}}$$

and

$$\cos\ (r_1 - a_1) = \pm \frac{1}{\sqrt{1 + \tan^2 a_1 \sin^2 \alpha_1 \tan^2 \varphi_1}}$$

From eqs. (52) we can also derive

$$\frac{d\vartheta_1}{d\varphi_1} \sin r_1 = \mp \frac{\sin a_1 \cos \alpha_1}{\sqrt{1 + \tan^2 a_1 \sin^2 \alpha_1 \tan^2 \varphi_1}} \quad [\text{compare eq. (57)}]$$

In the last three equations the upper signs are valid for the main curve and the lower signs for the antipode curve. If these expressions are substituted now in the above expression for $\frac{d\gamma_\nu}{d\vartheta_1}$, we obtain

$$d\gamma_\nu = -\frac{\sin a_1 \cos \alpha_1}{\tan a_1 \sin \alpha_1 \sin \varphi_1 \cos \varkappa_\nu\, a_1 \pm \sin \varkappa_\nu\, a_1}\, d\varphi_1$$

or, introducing

$$\left.\begin{aligned} \sin \alpha_\nu &= \frac{\tan a_1 \sin \alpha_1}{\tan \varkappa_\nu\, a_1} \\ \nu &= \frac{\cos \alpha_\nu \sin \varkappa_\nu\, a_1}{\sin a_1 \cos \alpha_1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (102)$$

$$d\gamma_\nu = \mp \frac{\cos \alpha_\nu}{\nu} \cdot \frac{d\varphi_1}{1 \pm \sin \alpha_\nu \sin \varphi_1}$$

This expression when integrated gives

$$\gamma_\nu = \pm \frac{2}{\nu} \arctan \left(\sqrt{\frac{1 \mp \sin \alpha_\nu}{1 \pm \sin \alpha_\nu}} \tan \frac{90^\circ - \varphi_1}{2} \right) + C$$

or, if the initial value for γ_ν is chosen so that $\gamma_\nu = 0$ for $\varphi_1 = 90^\circ$,

$$\tan \left(\pm \nu \frac{\gamma_\nu}{2} \right) = \tan \frac{90^\circ \mp \alpha_\nu}{2} \tan \frac{90^\circ - \varphi_1}{2}$$

where the upper signs correspond to the smaller angle between the axes of rotation $C_\nu = a_1(1 - \varkappa_\nu)$ and the lower signs correspond to the greater angle between the axes of rotation $C_\nu = a_1(1 + \varkappa_\nu)$. If the basic curve is rotated from $\vartheta_1 = 0$ to $\vartheta_1 = 180^\circ$, corresponding to $\varphi_1 = +90^\circ$ and $\varphi_1 = -90^\circ$, the mating curve will rotate, according to the above equation, through the angle $\frac{\pm 180^\circ}{\nu}$; ν is therefore the periodic number of the mating curve.

For the smaller distance between the axes of rotation we measure the polar angle from the point of contact where $\gamma_\nu = 0$, i.e. we put $\gamma_\nu = \vartheta_\nu$. For the greater distance between the axes of rotation we put $\gamma_\nu = \vartheta_\nu + \frac{180^\circ}{\nu}$. The previous equation then becomes

$$\tan \nu \frac{\vartheta_\nu}{2} = \tan \frac{90^\circ - \alpha_\nu}{2} \tan \frac{90^\circ \mp \varphi_1}{2} \quad \ldots\ldots\ldots\ldots\ldots \quad (103)$$

From the first equation in (7) and eq. (101) we obtain

$$r_\nu = r_1 - C_\nu = r_1 - a_1 (1 \mp \varkappa_\nu)$$

or

$$r_1 - a_1 = r_\nu \mp \varkappa_\nu a_1$$

which, together with the first equation in (52), gives

$$\tan (r_\nu \mp \varkappa_\nu a_1) = \tan a_1 \sin \alpha_1 \sin \varphi_1 \quad \ldots\ldots\ldots\ldots\ldots \quad (104)$$

with the additional condition that $(r_\nu \mp \varkappa_\nu a_1)$ and $(r_1 - a_1)$ for a given point of contact, i.e. for a given value of φ_1, must lie in the same quadrant (they are equal in magnitude). For the greater angle between the centres of rotation, where the lower sign is valid, we rotate the polar axis through 180° by changing the sign of r_ν. We then obtain

$$\tan (r_\nu - \varkappa_\nu a_1) = \pm \tan a_1 \sin \alpha_1 \sin \varphi_1 \quad \ldots\ldots\ldots\ldots\ldots \quad (105)$$

Finally, a new parameter φ_ν is introduced, defined by

$$\left.\begin{aligned} \varphi_\nu &= \varphi_1 \text{ for the smaller distance between the centres of rotation} \\ \varphi_\nu &= -\varphi_1 \text{ " " greater " " " " " " " "} \end{aligned}\right\} \quad \ldots\ldots \quad (106)$$

The rolling curve having the smaller angle between the centres of rotation, and the rolling curve having the greater angle between the centres of rotation, both then take the form

$$\left.\begin{aligned} &\tan (r_\nu - a_\nu) = \tan a_\nu \sin \alpha_\nu \sin \varphi_\nu = \tan a_1 \sin \alpha_1 \sin \varphi_\nu \\ &\tan \nu \frac{\vartheta_\nu}{2} = \tan \frac{90^\circ - \alpha_\nu}{2} \tan \frac{90^\circ - \varphi_\nu}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (107)$$

where

$$\left.\begin{aligned} \sin \alpha_\nu &= \frac{\tan a_1 \sin \alpha_1}{\tan a_\nu} \\ \nu &= \frac{\sin a_\nu \cos \alpha_\nu}{\sin a_1 \cos \alpha_1} \\ a_\nu &= \varkappa_\nu a_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (108)$$

where $(r_\nu - a_\nu)$ and $(r_1 - a_1)$ must lie in the same quadrant for the same values of φ_ν and φ_1. (This does not correspond to a point of contact for the greater angle between the centres of rotation.)

We always choose $a_\nu = \varkappa_\nu a_1 \leq 90^\circ$. A greater value of a_ν corresponds to an antipode curve to a curve having $\varkappa_\nu a_1 < 90^\circ$.

As for the basic curve, a main curve is defined as the curve obtained if $r_\nu - a_\nu$ is chosen in the first or fourth quadrant. From the above condition for the choice of $(r_\nu - a_\nu)$ it can be seen that, if the basic curve is a main curve, then the rolling curve must also be a main curve, and vice versa.

Eqs. (107) are valid only if $\tan a_1 \sin \alpha_1 < \tan \alpha_\nu$, because otherwise we should obtain $\sin \alpha_\nu > 1$. In this latter case we obtain, as will be shown later, a rolling curve with an imaginary value of ν.

If we put $\varkappa_\nu = 1$, we return to the basic curve, which is thus a twin rolling curve. It therefore follows that all its rolling curves are twin curves.

From eqs. (107) φ_ν may be eliminated. Eqs. (107) are in form the same as eqs. (52) if, in the latter, a_ν is substituted for a, α_ν for α, r_ν for r, $\nu\vartheta_\nu$ for ϑ and φ_ν for φ. It is therefore possible to eliminate φ_ν from eqs. (107) by simply making these substitutions in eq. (53), which is equivalent to eqs. (52). This gives

$$\tan r_\nu = \frac{\sin a_\nu \cos a_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu \left(\dfrac{1 - \sin^2 a_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu} - \cos \nu\vartheta_\nu \right)} \quad \ldots\ldots\ldots\ldots \quad (109)$$

Introducing the substitutions

$$\left.\begin{aligned} &\sin \alpha_\nu = \tan \frac{\alpha_T}{2} \quad (0 < \alpha_T < 90^\circ) \\ &\tan \frac{90^\circ + \alpha_T}{2} = \tan^2 \frac{90^\circ + \alpha_\nu}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (110)$$

or

and using the first equation in (108), the equation (109) may be rewritten in the form

$$\tan r_\nu = \frac{\sin 2 a_\nu}{\tan \alpha_T \left(\dfrac{1 - \sin^2 a_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu} - \cos \nu\vartheta_\nu \right)} \quad \ldots\ldots\ldots\ldots \quad (111)$$

or

$$\left.\begin{aligned} \tan r_\nu &= \frac{k_1}{k_2 - \cos \nu\vartheta_\nu} \\ \text{where} \\ k_1 &= \frac{\sin 2\, a_\nu}{\tan \alpha_T} = \frac{\sin 2\, \alpha_\nu\, a_1}{\tan \alpha_T} \\ k_2 &= \frac{\cos^2 a_\nu}{\tan \frac{\alpha_T}{2}} + \sin^2 a_\nu \tan \frac{\alpha_T}{2} = \frac{1 + \cos 2\, a_\nu \cos \alpha_T}{\sin \alpha_T} \\ \tan \frac{\alpha_T}{2} &= \frac{\tan a_1 \sin \alpha_1}{\tan a_\nu} = \sin \alpha_\nu \\ \nu &= \frac{\sin a_\nu \cos \alpha_\nu}{\sin a_1 \cos \alpha_1} \end{aligned}\right\}$$

k_1 and k_2 are positive.

From the second and third equations in this system of equations we obtain

$$\tan 2\, a_\nu = \frac{k_1 \sin \alpha_T}{k_2 \sin \alpha_T - 1}$$

and

$$1 = \sin^2 2\, a_\nu + \cos^2 2\, a_\nu = \frac{k_1^2 \sin^2 \alpha_T + (k_2 \sin \alpha_1 - 1)^2}{\cos^2 \alpha_T}$$

or, if we solve with respect to $\sin \alpha_T$,

$$\sin \alpha_T = \frac{2\, k_2}{k_2^2 + k_1^2 + 1}$$

which, substituted in the above expression for $\tan 2\, a_\nu$, gives

$$\tan 2\, a_\nu = \frac{2\, k_1\, k_2}{k_2^2 - k_1^2 - 1}$$

The last two equations in the above system of equations may be rewritten in the form

$$\left.\begin{aligned} \tan a_1 &= \frac{\tan a_\nu \sin \alpha_\nu}{\sin \alpha_1} \\ \sin a_1 &= \frac{\sin a_\nu \cos \alpha_\nu}{\nu \cos \alpha_1} \end{aligned}\right\}$$

from which

$$\cos a_1 = \frac{\cos a_\nu \tan \alpha_1}{\nu \tan \alpha_\nu}$$

Squaring and adding $\sin a_1$ and $\cos a_1$ will give

$$\nu^2 \cos^2 \alpha_1 \tan^2 \alpha_\nu = \sin^2 a_\nu \sin^2 \alpha_\nu + \cos^2 a_\nu \sin^2 \alpha_1$$

or, if we solve with respect to $\cos \alpha_1$,

$$\cos \alpha_1 = \cos \alpha_\nu \sqrt{\frac{\cos^2 a_\nu + \tan^2 \alpha_\nu}{\cos^2 a_\nu + \nu^2 \tan^2 \alpha_\nu}}$$

The above system of equations can now be rewritten in the form

$$\left.\begin{aligned}
&\tan \tau_\nu = \frac{k_1}{k_2 - \cos \nu\vartheta_\nu} \quad (k_1 \text{ and } k_2 \text{ are positive})\\
&\tan 2\, a_\nu = \frac{2\, k_1 k_2}{k_2^2 - k_1^2 - 1}\\
&\sin \alpha_T = \frac{2\, k_2}{k_2^2 + k_1^2 + 1}\\
&\sin \alpha_\nu = \tan \frac{\alpha_T}{2}\\
&\cos \alpha_1 = \cos \alpha_\nu \sqrt{\frac{\cos^2 a_\nu + \tan^2 \alpha_\nu}{\cos^2 a_\nu + \nu^2 \tan^2 \alpha_\nu}}\\
&\sin a_1 = \frac{\sin a_\nu \cos \alpha_\nu}{\nu \cos \alpha_1} \text{ or } \tan a_1 = \frac{\tan a_\nu \sin \alpha_\nu}{\sin \alpha_1}\\
&(\tan a_1 \sin \alpha_1 = \tan a_\nu \sin \alpha_\nu = \tan e a_1)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(112)}$$

If $a_\nu = \varkappa_\nu\, a_1 \to 90°$, then $k_2 \to 0$.

This system of equations is used if we have an equation given in the same form as the first equation in the system. By using the following equations, it is then possible to determine α_ν, $a_\nu = \varkappa_\nu\, a_1$, α_1 and a_1, after which the equation may be expressed in the forms (107) or (109).

We always choose α_T, α_ν, α_1 and $a_\nu = \varkappa_\nu\, a_1$ in the first quadrant.

The condition for obtaining real solutions is that $\cos \alpha_1$ is positive and ≤ 1, i.e.

$$\cos^2 a_\nu \cos^2 \alpha_\nu + \sin^2 \alpha_\nu \leq \cos^2 a_\nu + \nu^2 \tan^2 \alpha_\nu$$

or

$$\nu \geq \sin a_\nu \cos \alpha_\nu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(113)}$$

Then not only will α_1 be real, but also a_1, which can be seen from the expression for $\sin a_1$ in eqs. (112).

If the condition (113) is not fulfilled, the given curve is not a rolling curve to the real spherical conic section but instead, as will be shown later, a rolling curve to the imaginary spherical hyperbola.

The limiting case where $\nu = \cos \alpha_\nu \sin a_\nu$, may be considered as a rolling curve both to the real spherical conic section and to the imaginary spherical hyperbola. In the following, however, it will be treated as a rolling curve to the real spherical conic section.

In eqs. (112) we must always have $\sin \alpha_T \leq 1$. This is always the case, however, as may be found if we derive the max. value of $\sin \alpha_T$.

Example

According to eqs. (107) it is possible to construct spherical conic section rolling curves with real ν-values for gears with an arbitrary shaft angle, an arbitrary mean gear ratio and an arbitrary ratio between the max. and min. momentary gear ratio (arbitrary α-value). As an example, we shall now construct a rolling curve pair with external meshing, having a mean gear ratio equal to $\frac{2}{1}$ in which the one curve, the L-curve, is a real spherical conic

section curve having $\alpha_1 = 30°$. The angle between the centres of rotation is assumed to be 90°.

As for the spherical conic section curve (the basic curve) $\nu_L = 1$, we obtain for the mating curve $\nu_R = 2$. If these values are substituted in eqs. (108), we obtain

$$\left.\begin{aligned} \sin \alpha_2 &= \frac{\tan a_1 \sin 30°}{\tan a_2} \\ \nu_R &= \frac{\sin a_2 \cos \alpha_2}{\sin a_1 \cos 30°} = 2 \end{aligned}\right\}$$

Furthermore, the angle between the centres of rotation equals

$$C_\nu = C_2 = a_1 (1 + \varkappa_2) = 90°$$

from which

$$a_2 = \varkappa_2 a_1 = 90° - a_1$$

where the suffixes 1 and 2 indicate whether the values belong to the single-period or the two-period curve.

If these expressions are substituted in the above system of equations, we obtain

$$\left.\begin{aligned} \sin \alpha_2 &= \tan^2 a_1 \sin 30° \\ \tan a_1 &= \frac{1}{2} \cdot \frac{\cos \alpha_2}{\cos 30°} \end{aligned}\right\}$$

from which

$$2 \sin \alpha_2 = \frac{1}{3} \cos^2 \alpha_2$$

If we solve this expression with respect to $\sin \alpha_2$, we obtain $\sin \alpha_2 = -3 + \sqrt{10} = 0.16228$ thus $\alpha_2 = 9.339°$.

We now obtain from the above expressions

$$\left.\begin{aligned} &\tan a_1 = 0.56969, \text{ thus } a_1 = 29.670° \\ &a_2 = \varkappa_2 a_1 = 90° - a_1 = 60.330° \end{aligned}\right\}$$

The equations of the required rolling curves are thus according to eqs. (107):

(a) The equation of the L-curve (the basic curve)

$$\left.\begin{aligned} &\tan (r_1 - 29.670°) = \tan 29.670° \sin 30° \sin \varphi_1 = 0.28485 \sin \varphi_1 \\ &\tan \frac{\vartheta_1}{2} = \tan \frac{90° - 30°}{2} \tan \frac{90° - \varphi_1}{2} = 0.57735 \tan \frac{90° - \varphi_1}{2} \end{aligned}\right\}$$

(b) The equation of the R-curve (the two-period curve)

$$\left.\begin{aligned} &\tan (r_2 - 60.330°) = \tan 29.670° \sin 30° \sin \varphi_2 = 0.28485 \sin \varphi_2 \\ &\tan \vartheta_2 = \tan \frac{90° - 9.339°}{2} \tan \frac{90° - \varphi_2}{2} = 0.84898 \tan \frac{90° - \varphi_2}{2} \end{aligned}\right\}$$

According to (109) the equation of the L-curve may also be written in the form

$$\tan r_1 = \frac{\sin 29.670^\circ \cos 29.670^\circ \cos^2 30^\circ}{\sin 30^\circ \left(\frac{1 - \sin^2 29.670^\circ \cos^2 30^\circ}{\sin 30^\circ} - \cos \vartheta_1 \right)} = \frac{0.64516}{1.63245 - \cos \vartheta_1}$$

and the equation of the R-curve in the form

$$\tan r_2 = \frac{\sin 60.330^\circ \cos 60.330^\circ \cos^2 9.339^\circ}{\sin 9.339^\circ \left(\frac{1 - \sin^2 60.330^\circ \cos^2 9.339^\circ}{\sin 9.339^\circ} - \cos 2\vartheta_2 \right)} = \frac{2.58056}{1.63242 - \cos 2\vartheta_2}$$

According to the foregoing not only the main curves of the L-curve and R-curve can form a rolling curve pair with the angle between the centres of rotation $= 90^\circ$, but also two L-curves can form a rolling curve pair having an angle between the centres of rotation $= 2a_1 = 59.340^\circ$, and two R-curves can form a rolling curve pair having an angle between the centres of rotation $= 2\varkappa_2 a_1 = 120.660^\circ$. Furthermore, the L-curve can form, together with the R-curve, a rolling curve pair having internal meshing with the angle between the centres of rotation $= |a_1 - \varkappa_2 a_1| = 30.660^\circ$. If in the latter case the R-curve is replaced by its antipode curve, we obtain a rolling curve pair having external meshing and an angle between the centres of rotation $= 180^\circ - 30.660^\circ = 149.340^\circ$.

The rolling curves computed above are shown in Figs. 6 and 7a.

2) Rolling curves with imaginary periodic number

If in eqs. (108) $\tan a_\nu < \tan a_1 \sin \alpha_1 = \tan e\, a_1$, i.e. $\varkappa_\nu < e$, then $\sin \alpha_\nu > 1$ and $\cos \alpha_\nu = \sqrt{1 - \sin^2 \alpha_\nu}$ will be purely imaginary. We then make the substitutions

$$\left.\begin{aligned} \sin \alpha_\nu &= \frac{1}{\sin \bar{\alpha}_\nu} \\ \cos \alpha_\nu &= \frac{i}{\tan \bar{\alpha}_\nu} \\ \nu &= i\,|\nu| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (114)$$

Eqs. (107) and (108) then become

$$\left.\begin{aligned} \tan (r_\nu - a_\nu) &= \frac{\tan a_\nu}{\sin \bar{\alpha}_\nu} \sin \varphi_\nu \\ \tanh |\nu| \frac{\vartheta_\nu}{2} &= \tan \frac{90^\circ - \bar{\alpha}_\nu}{2} \tan \frac{90^\circ - \varphi_\nu}{2} \end{aligned}\right\}$$

where

$$\left.\begin{aligned} \sin \bar{\alpha}_\nu &= \frac{\tan a_\nu}{\tan a_1 \sin \alpha_1} \\ |\nu| &= \frac{\sin a_\nu}{\sin a_1 \cos \alpha_1 \tan \bar{\alpha}_\nu} \\ a_\nu &= \varkappa_\nu a_1 \end{aligned}\right\}$$

These equations constitute a rolling curve to the real spherical conic section with imaginary $\nu = i|\nu|$. The equation in non-parametric form is obtained by inserting the foregoing substitutions (114) in (109), which gives

$$\tan r_\nu = -\frac{\sin a_\nu \cos a_\nu \cos^2 \bar{\alpha}_\nu}{\sin \bar{\alpha}_\nu \left(\frac{1 - \cos^2 a_\nu \sin^2 \bar{\alpha}_\nu}{\sin \bar{\alpha}_\nu} - \cosh |\nu| \bar{\vartheta}_\nu \right)}$$

It would appear that the above system of equations has a real solution only if

$$\left| \tan \frac{90^\circ - \bar{\alpha}_\nu}{2} \tan \frac{90^\circ - \varphi_\nu}{2} \right| < 1$$

or

$$\tan \frac{90^\circ - \bar{\alpha}_\nu}{2} < \left| \tan \frac{90^\circ + \varphi_\nu}{2} \right|$$

or

$$-\bar{\alpha}_\nu < \varphi_\nu < 180^\circ + \bar{\alpha}_\nu$$

because otherwise ϑ_ν would be complex. If, however, $-\alpha_\nu > \varphi_\nu > 180^\circ + \bar{\alpha}_\nu$, a solution can be obtained by rotating the polar axis through the complex angle $i \frac{180^\circ}{|\nu|}$, i.e. by introducing the substitution

$$\vartheta = \bar{\vartheta}_\nu - i \frac{180^\circ}{|\nu|}$$

We then obtain

$$\tanh |\nu| \frac{\vartheta_\nu}{2} = \frac{1}{i} \tan i |\nu| \frac{\vartheta_\nu}{2} = \frac{1}{i} \tan \left(i |\nu| \frac{\bar{\vartheta}_\nu}{2} + 90^\circ \right) = -\frac{1}{i \tan i |\nu| \frac{\bar{\vartheta}_\nu}{2}} = \frac{1}{\tanh |\nu| \frac{\bar{\vartheta}_\nu}{2}}$$

and

$$\cosh |\nu| \vartheta_\nu = \cos i |\nu| \vartheta_\nu = \cos (i |\nu| \bar{\vartheta}_\nu + 180^\circ) = -\cos i |\nu| \bar{\vartheta}_\nu = -\cosh |\nu| \vartheta_\nu$$

Substituting this in the second of the equations for the curve in parametric form we obtain

$$\tanh |\nu| \frac{\bar{\vartheta}_\nu}{2} = \tan \frac{90^\circ + \bar{\alpha}_\nu}{2} \tan \frac{90^\circ + \varphi_\nu}{2}$$

If the expressions are substituted in the non-parametric equation, we obtain the same equation again with the difference that $\cosh |\nu| \vartheta_\nu$ is replaced by $-\cosh |\nu| \bar{\vartheta}_\nu$. Thus, if the bars in $\bar{\alpha}_\nu$ and $\bar{\vartheta}_\nu$ are omitted, both cases can be combined in the following equations:

The equation of the curve in parametric form

$$\left.\begin{aligned} \tan (r_\nu - a_\nu) &= \frac{\tan a_\nu}{\sin \alpha_\nu} \sin \varphi_\nu = \tan a_1 \sin \alpha_1 \sin \varphi_\nu \\ \tanh |\nu| \frac{\vartheta_\nu}{2} &= \tan \frac{90^\circ \mp \alpha_\nu}{2} \tan \frac{90^\circ \mp \varphi_\nu}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (115)$$

where

$$\left.\begin{aligned} \sin \alpha_\nu &= \frac{\tan a_\nu}{\tan a_1 \sin \alpha_1} \\ |\nu| &= \frac{\sin a_\nu}{\sin a_1 \cos \alpha_1 \tan \alpha_\nu} \\ a_\nu &= \varkappa_\nu a_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (116)$$

and the equation in non-parametric form

$$\tan r_\nu = -\frac{\sin a_\nu \cos a_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu \left(\frac{1 - \cos^2 a_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu} \mp \cosh |\nu| \vartheta_\nu \right)} \quad \ldots\ldots\ldots \quad (117)$$

In equations (115) and (117) $-\alpha_\nu < \varphi_\nu < 180° + \alpha_\nu$ corresponds to the upper sign (otherwise ϑ_ν would be imaginary) and $-\alpha_\nu > \varphi_\nu > 180° + \alpha_\nu$ corresponds to the lower sign.

The first case gives a larger branch which approaches the origin in a spiral, and this curve is called the larger branch. The latter case gives a similar, but smaller, branch, and this curve is designated as the smaller branch.

Because the curves with imaginary ν-values just considered are rolling curves to the real spherical conic section, which is a twin curve, these curves are also twin curves.

If we introduce also for rolling curves of this type the substitution (110)

$$\sin \alpha_\nu = \tan \frac{\alpha_T}{2}$$

and substitution (101)

$$a_\nu = \varkappa_\nu a_1$$

eq. (117) may be transformed in the same way as eq. (109) was transformed into (112).

We obtain

$$\left.\begin{aligned}
&\tan r_\nu = \frac{k_1}{k_2 - \cosh |\nu| \vartheta_\nu} = \frac{\mp |k_1|}{\pm |k_2| - \cosh |\nu| \vartheta_\nu} \\
&\tan 2 a_\nu = -\frac{2 k_1 k_2}{k_1^2 - k_2^2 + 1} = \frac{2 |k_1| |k_2|}{k_1^2 - k_2^2 + 1} \\
&\sin \alpha_T = \frac{2 |k_2|}{k_2^2 + k_1^2 + 1} \\
&\sin \alpha_\nu = \tan \frac{\alpha_T}{2} \\
&\cos \alpha_1 = \frac{1}{\tan \alpha_\nu} \sqrt{\frac{1 - \cos^2 a_\nu \cos^2 \alpha_\nu}{|\nu|^2 + \cos^2 a_\nu \cos^2 \alpha_\nu}} \\
&\sin a_1 = \frac{\sin a_\nu}{|\nu| \cos \alpha_1 \tan \alpha_\nu} \text{ or } \tan a_1 = \frac{\tan a_\nu}{\sin \alpha_1 \sin \alpha_\nu} \\
&\left(\tan a_1 \sin \alpha_1 = \frac{\tan a_\nu}{\sin \alpha_\nu} = \tan e a_1 \right)
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (118)$$

If $|k_2| \to 0$ then $a_\nu \to 0$.

By comparing this system of equations with eq.! (117) it can be seen that we must have in the former

either k_1 negative and k_2 positive, which corresponds to the larger branch and the upper signs in the second equation in (115) and in (117),

or k_1 positive and k_2 negative, which corresponds to the smaller branch and the lower signs in the second equation in (115) and in (117).

We always choose α_T, α_ν, α_1 and $\varkappa_\nu$ a_1 in the first quadrant.

The condition for obtaining real solutions is that $\cos \alpha_1$ is positive and ≤ 1, i.e.

$$1 - \cos^2 a_\nu \cos^2 \alpha_\nu \leq |\nu|^2 \tan^2 \alpha_\nu + \cos^2 a_\nu \sin^2 \alpha_\nu$$

or

$$|\nu| \geq \frac{\sin a_\nu}{\tan \alpha_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (119)$$

In that case not only will α_1 be real, but also a_1, which can be seen from the sixth equation in (118). If condition (119) is not fulfilled, the curve is not a rolling curve to the real spherical conic section, but instead, as will be shown later on, a rolling curve with imaginary ν-value to the imaginary spherical hyperbola.

α_T will always be real, which can be shown by deriving the max. value of $\sin \alpha_T$.

The case where $|\nu| = \dfrac{\sin a_\nu}{\tan \alpha_\nu}$ is a limiting case between rolling curves with imaginary ν-values to the real spherical conic section and to the imaginary spherical hyperbola. By using the system of equations (118) we then obtain $\alpha_1 = 0$ and $a_1 = 90°$.

3) Special cases of rolling curves to the real spherical conic section

α) The crown-wheel curve (the spherical cos-curve having $\nu \geq 1$)

As the crown-wheel curve we define the rolling curve having $\varkappa_\nu a_1 = 90°$, i.e. having the arithmetic mean radius vector $= 90°$.

If we put $a_\nu = \varkappa_\nu a_1 = 90°$, we obtain at first, according to the relations between a_1, α_1, α_ν, ν and $\varkappa_\nu a_1$ in the two last equations in (108),

$$\left.\begin{aligned} &\sin \alpha_\nu = 0, \text{ thus } \alpha_\nu = 0 \\ &\nu = \frac{1}{\sin a_1 \cos \alpha_1} \end{aligned}\right\}$$

Eqs. (107) then take the form

$$\left.\begin{aligned} &\tan (r_\nu - 90°) = \tan a_1 \sin \alpha_1 \sin \varphi_\nu \\ &\nu \vartheta_\nu = 90° - \varphi_\nu \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (120)$$

where

$$\left.\begin{aligned} &\alpha_\nu = 0 \\ &\nu = \frac{1}{\sin a_1 \cos \alpha_1} \\ &a_\nu = \varkappa_\nu a_1 = 90° \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (121)$$

which is the equation of the crown-wheel curve in parametric form.

The equation in non-parametric form may be obtained by substituting the above expression for α_ν and the first equation of (108) in eqs. (109)

$$\tan r_\nu = -\frac{1}{\tan a_1 \sin \alpha_1 \cos \nu \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots \quad (122)$$

The equation of this type of curve is included in the system of equations (112) in which, however, $k_2 = 0$. If k_1 and k_2 $(= 0)$ are given, it must be remembered, when treating the

system of equations (112), that since k_2 is positive ($= +0$) then $2\varkappa_\nu a_1 = 180°$. Thus $\varkappa_\nu a_1 = 90°$, which is correct.

The fifth equation in (112) becomes indeterminate. Instead of this equation, however, it is possible to use the relation $k_1 = \dfrac{1}{\tan a_1 \sin \alpha_1}$ obtained direct from eq. (122). If we apply the sixth equation in (112), which becomes $\sin a_1 = \dfrac{1}{\nu \cos \alpha_1}$, a_1 can be eliminated and we obtain

$$\cos \alpha_1 = \sqrt{\frac{1 + k_1^2}{\nu^2 + k_1^2}}$$

For the crown-wheel curve, therefore, the system of equations (112) takes the form

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\cos \nu\vartheta_\nu} \ (k_1 \text{ is positive}) \\ &a_\nu = 90° \\ &\alpha_\nu = 0 \\ &\cos \alpha_1 = \sqrt{\frac{1 + k_1^2}{\nu^2 + k_1^2}} \\ &\sin a_1 = \frac{1}{\nu \cos \alpha_1} \\ &\left(\tan a_1 \sin \alpha_1 = \tan ea_1 = \frac{1}{k_1}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (123)$$

If we project radially the curve on to a cylinder tangential to the great circle $r = 90°$ by means of radii from the centre of the sphere and if we place the x-axis on this great circle, and the y-axis along a generatrix of the cylinder, the equation of the projected curve can be written in the form

$$\frac{y}{\tan a_1 \sin \alpha_1} = -\cos \nu\vartheta_\nu = -\cos \frac{x}{\sin a_1 \cos \alpha_1}$$

because $y = \tan (90° - r)$.

For this reason the curve is named the spherical cos-curve.

The condition for obtaining real solutions (113) becomes

$$\nu \geq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (124)$$

Eqs. (120) and (122) therefore do not include all spherical cos-curves, but only those where $\nu = \dfrac{1}{\sin a_1 \cos \alpha_1} \geq 1$. It will be shown later on that the remaining type of curve having $\nu < 1$ are the crown-wheel rolling curves to the imaginary spherical hyperbola.

In the limiting case where $\nu = 1$ (the limiting case to rolling curves to the imaginary spherical hyperbola) we obtain $\alpha_1 = 0$ and $a_1 = 90°$. When using eq. (122) we must then observe that $\lim (\tan a_1 \sin \alpha_1) = \dfrac{1}{k_1}$.

β) The rolling curve with $\varkappa_\nu$ $a_1 = a_\nu = 0$ (*the spherical cosh-curve having* $|\nu| \geq |\tan r_{\nu\max}|$)

If $a_\nu = \varkappa_\nu$ $a_1 = 0$, we obtain from the system of equations (116)

$$\sin \alpha_\nu = \frac{\tan a_\nu}{\tan a_1 \sin \alpha_1}, \text{ thus } \alpha_\nu = 0$$

and

$$|\nu| = \frac{\sin a_\nu}{\sin a_1 \cos \alpha_1 \tan \alpha_\nu} = \frac{\sin a_\nu \sin \alpha_\nu \tan a_1 \sin \alpha_1}{\sin a_1 \cos \alpha_1 \tan \alpha_\nu \tan a_\nu} = \frac{\cos a_\nu \cos \alpha_\nu \tan \alpha_1}{\cos a_1} \text{ or } |\nu| = \frac{\tan \alpha_1}{\cos a_1}$$

The equation of the rolling curve in parametric form thus becomes

$$\left.\begin{aligned} &\tan r_\nu = \tan a_1 \sin \alpha_1 \sin \varphi_\nu \\ &\tanh |\nu| \frac{\vartheta_\nu}{2} = \tan \frac{90° \mp \varphi_\nu}{2} \text{ or } |\nu| \vartheta_\nu = \frac{180°}{\pi} \ln \tan \left(90° \mp \frac{\varphi_\nu}{2}\right) \end{aligned}\right\} \quad \ldots\ldots \quad (125)$$

where

$$\left.\begin{aligned} &\alpha_\nu = 0 \\ &|\nu| = \frac{\tan \alpha_1}{\cos a_1} \\ &a_\nu = \varkappa_\nu \; a_1 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (126)$$

The equation in non-parametric form is obtained by substituting the first equation of (116) and the above values of α_ν and $|\nu|$ in eq. (117), this giving

$$\tan r_\nu = \pm \frac{\tan a_1 \sin \alpha_1}{\cosh |\nu| \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (127)$$

where the upper sign corresponds to the larger, and the lower sign to the smaller, branch of normal rolling curves. In this limiting case, however, both the branches have the same size.

The equation of this type of curve is also included in the system of equations (118) if $|k_2| = 0$. If in (118) $|k_2| \to 0$, then $\alpha_\nu \to 0$ and $a_\nu \to 0$, in accordance with the assumption above. The fifth equation in (118) becomes indeterminate, but instead of this equation we can use the relation $|k_1| = \tan a_1 \sin \alpha_1$, which is obtained direct from eq. (127). The sixth equation can be rewritten in the form

$$\cos a_1 = \frac{\tan \alpha_1 \cos a_\nu \cos \alpha_\nu}{|\nu|}$$

For this special case eqs. (118) now become

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\cos |\nu| \vartheta_\nu} \\ &a_\nu = \varkappa_\nu \; a_1 = 0 \text{ and } \alpha_\nu = 0 \\ &|k_1| = \tan a_1 \sin \alpha_1 \\ &\cos a_1 = \frac{\tan \alpha_1}{|\nu|} \end{aligned}\right\}$$

Eliminating a_1 from the last two equations, we obtain

$$\cos \alpha_1 = \sqrt{\frac{k_1^2 + 1}{|\nu|^2 + 1}} \text{ or } \tan \alpha_1 = \sqrt{\frac{|\nu|^2 - k_1^2}{1 + k_1^2}}$$

The above system of equations can now be rewritten in the form

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\cosh |\nu| \vartheta_\nu} = \frac{\mp |k_1|}{-\cosh |\nu| \vartheta_\nu} \\ &a_\nu = 0 \\ &\alpha_\nu = 0 \\ &\tan \alpha_1 = \sqrt{\frac{|\nu|^2 - k_1^2}{1 + k_1^2}} \text{ or } \cos \alpha_1 = \sqrt{\frac{k_1^2 + 1}{|\nu|^2 + 1}} \\ &\cos a_1 = \frac{\tan \alpha_1}{|\nu|} = \sqrt{\frac{1 - \frac{k_1^2}{|\nu|^2}}{1 + k_1^2}} \\ &(\tan a_1 \sin \alpha_1 = \tan e\, a_1 = |k_1|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (128)$$

By using the first equation in (116), the condition for obtaining real solutions (119) can be rewritten as

$$|\nu| \geq \tan a_1 \sin \alpha_1 = \tan e a_1 = |\tan r_{\nu \max}| = |k_1| \quad \ldots\ldots\ldots\ldots \quad (129)$$

In the limiting case where $|\nu| = |k_1|$ we obtain $a_1 = 90°$ and $\alpha_1 = 0$. When using eqs. (125) and (127) it must then be observed that $\lim (\tan a_1 \sin \alpha_1) = |k_1|$.

If the curve is projected radially from the centre of the sphere on to a cylinder, tangential to the great circle $r = 90°$, and if we place the x-axis on this circle and the y-axis along a generatrix of the cylinder, the equation of the projected curve takes the form

$$y \tan a_1 \sin \alpha_1 = \pm \cosh |\nu| \vartheta_\nu = \pm \cosh \left(\frac{\tan \alpha_1}{\cos a_1} x \right)$$

For this reason, therefore, the curve is designated the spherical cosh-curve.

From the condition (129) it can be understood that equations (125) and (127) do not include all spherical cosh-curves, but only those where $|\nu| \geq \tan a_1 \sin \alpha_1 = |\tan r_{\nu \max}|$. Later it will be shown that the remaining curves, where $|\nu| < |\tan r_{\nu \max}|$, are rolling curves to the imaginary spherical hyperbola having $\varkappa_\nu = 0$.

γ) The rolling curve with periodic number $\nu = 0$

If in the second equation in (108) we insert $\nu = 0$, we obtain $\cos \alpha_0 \sin a_0 = 0$ or, since $a_0 = \varkappa_0 a_1 \neq 0$,

$$\cos \alpha_0 = 0, \text{ thus } \alpha_0 = 90° \text{ and } \sin \alpha_0 = 1$$

The first equation in (108) and eq. (46) now give

$$\tan a_0 = \tan a_1 \sin \alpha_1 = \tan e\, a_1$$

thus

$$\varkappa_0 = e$$

The second equation in (107) can be rewritten in the form

$$\tan \nu \frac{\vartheta_0}{2} = \frac{1 - \sin \alpha_0}{\cos \alpha_0} \tan \frac{90^\circ - \varphi_0}{2}$$

or, since $\nu\vartheta_0 \to 0$ if $\nu \to 0$

$$\vartheta_0 = \frac{2\,(1 - \sin \alpha_0)}{\nu \cos \alpha_0} \tan \frac{90^\circ - \varphi_0}{2}$$

Substituting ν from the second equation in (108) gives

$$\vartheta_0 = \frac{2\,(1 - \sin \alpha_0) \sin a_1 \cos \alpha_1}{\sin a_0 \cos^2 \alpha_0} \tan \frac{90^\circ - \varphi_0}{2} = \frac{2 \sin a_1 \cos \alpha_1}{\sin a_0\,(1 + \sin \alpha_0)} \tan \frac{90^\circ - \varphi_0}{2}$$

or, since $\sin \alpha_0 \to 1$,

$$\vartheta_0 = \frac{\sin a_1 \cos \alpha_1}{\sin a_0} \tan \frac{90^\circ - \varphi_0}{2}$$

The equation of the rolling curve in parametric form therefore becomes

$$\left.\begin{aligned} \tan (r_0 - a_0) &= \tan a_0 \sin \varphi_0 = \tan a_1 \sin \alpha_1 \sin \varphi_0 \\ \vartheta_0 &= \frac{\sin a_1 \cos \alpha_1}{\sin a_0} \tan \frac{90^\circ - \varphi_0}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (130)$$

where

$$\left.\begin{aligned} \alpha_0 &= 90^\circ \\ \nu &= 0 \\ a_0 &= \varkappa_0 a_1 = e\, a_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (131)$$

ϑ_0 is measured in radians.

The equation of the rolling curve in non-parametric form may be obtained by eliminating φ_0 from (130) in the following way.

From the first equation we solve with respect to $\sin \varphi_0$

$$\sin \varphi_0 = \frac{\tan r_0 - \tan a_0}{\tan a_0\,(1 + \tan r_0 \tan a_0)}$$

and form

$$\tan^2 \frac{90^\circ - \varphi_0}{2} = \frac{1 - \sin \varphi_0}{1 + \sin \varphi_0} = \cos^2 a_0 \frac{2 \tan a_0 - \tan r_0\,(1 - \tan^2 a_0)}{\tan r_0}$$

which substituted in the second equation of (130) gives

$$\vartheta_0^2 = \frac{\sin^2 a_1 \cos^2 \alpha_1\,[2 \tan a_0 - \tan r_0\,(1 - \tan^2 a_0)]}{\tan^2 a_0 \tan r_0}$$

and if we solve with respect to $\tan r_0$ and use eq. (131), we obtain the equation in non-parametric form

$$\tan r_0 = \frac{2 \tan a_0}{(1 - \tan^2 a_0) + \dfrac{\tan^2 a_0 + \sin^2 \alpha_1}{\cos^2 \alpha_1} \vartheta_0^2} \quad \ldots\ldots\ldots\ldots \quad (132)$$

where ϑ_0 is measured in radians.

The equation for this type of curve may also be derived by starting from the equation for the normal rolling curves with imaginary ν-values, and putting $|\nu| = 0$. The curve may therefore be regarded as a limiting case between the rolling curves with real and with imaginary ν-values.

The equation in non-parametric form may also be written

where

$$\left.\begin{aligned} &\tan r_0 = \frac{k_1}{k_2 + \vartheta_0^2} \\ &k_1 = 2 \tan a_0 \frac{\cos^2 \alpha_1}{\tan^2 a_0 + \sin^2 \alpha_1} \\ &k_2 = (1 - \tan^2 a_0) \frac{\cos^2 \alpha_1}{\tan^2 a_0 + \sin^2 \alpha_1} \\ &\tan a_0 = \tan \varkappa_0 a_1 = \tan a_1 \sin \alpha_1 = \tan e a_1 \end{aligned}\right\}$$

Dividing the second equation by the third gives

$$\frac{k_1}{k_2} = \frac{2 \tan a_0}{1 - \tan^2 a_0} = \tan 2 a_0$$

From the second equation we can solve with respect to $\cos \alpha_1$, thereby obtaining

$$\cos \alpha_1 = \frac{1}{\cos a_0} \sqrt{\frac{k_1}{2 \tan a_0 + k_1}} \text{ or } \sin \alpha_1 = \tan a_0 \sqrt{\frac{\frac{2}{\tan a_0} - k_1}{2 \tan a_0 + k_1}}$$

The above system of equations may now be written in the form

$$\left.\begin{aligned} &\tan r_0 = \frac{k_1}{k_2 + \vartheta_0^2} \ (k_1 \text{ is positive}) \\ &\tan 2 a_0 = \frac{k_1}{k_2} \\ &\alpha_0 = 90^\circ \\ &\cos \alpha_1 = \frac{1}{\cos a_0} \sqrt{\frac{k_1}{2 \tan a_0 + k_1}} \text{ or } \sin \alpha_1 = \tan a_0 \sqrt{\frac{\frac{2}{\tan a_0} - k_1}{2 \tan a_0 + k_1}} \\ &\tan a_1 = \frac{\tan a_0}{\sin \alpha_1} \\ &(\tan a_1 \sin \alpha_1 = \tan a_0 = \tan e a_1) \end{aligned}\right\} \quad .. \quad (133)$$

where ϑ_0 is measured in radians.

This system of equations is used if the equation of the rolling curve is given in the same form as the first equation. By using the following equations, α_1 and a_1 may be determined, after which the equation can be written down in any of the forms (130) or (132). As for the normal rolling curves, $a_0 = \varkappa_0 a_1$, a_1 and α_1 are chosen in the first quadrant.

The condition for obtaining real solutions is that $\cos \alpha_1$ is real and ≤ 1. Since k_1 is positive, we must have

$$k_1 \leq k_1 \cos^2 a_0 + 2 \sin a_0 \cos a_0$$

or

$$\frac{2}{k_1} \geq \tan a_0$$

If this condition is not fulfilled, the curve is not a rolling curve to the real spherical conic section, but, as will be shown later, a rolling curve to the imaginary spherical hyperbola, where $\nu \to 0$.

If we form $\tan 2\, a_0$ from this expression, we obtain by using the second equation in (133)

$$\tan 2\, a_0 = \frac{k_1}{k_2} \leq \frac{4/k_1}{1 - 4/k_1^2} \text{ if } k_2 \text{ and } (1 - 4/k_1^2) \text{ have the same signs}$$

or

$$\frac{k_1}{k_2} \geq \frac{4/k_1}{1 - 4/k_1^2} \text{ if } k_2 \text{ and } (1 - 4/k_1^2) \text{ have different signs}$$

In both cases the above condition can now be rewritten in the form

$$\frac{k_1^2}{4} - k_2 \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (134)$$

In the limiting case where $\tan a_0 = \frac{2}{k_1}$ the above system of equations gives $\alpha_1 = 0$ and $a_1 = 90°$.

δ) The curve having $e = 0$ (the spherical circle)

If $e = \sin \alpha_1 = 0$, thus $\alpha_1 = 0$ and $\alpha_\nu = 0$, then eqs. (107) take the form

$$\left.\begin{aligned} r_\nu &= a_\nu = \varkappa_\nu\, a_1 \\ \nu\vartheta_\nu &= 90° - \varphi_\nu \end{aligned}\right\}$$

which is a spherical circle.

The first equation in (108) then gives $\alpha_\nu = 0$. The fourth equation makes ν indeterminate, and we can choose any value of ν for one of the curves in the rolling curve pair, but the relation between ν_L and ν_R is given by the second equation in (108).

$$\frac{\nu_L}{\nu_R} = \frac{\sin a_{\nu L}}{\sin a_{\nu R}}$$

It is, however, convenient to choose $\nu_L = 1$ for the great circle. We then obtain $a_1 = 90°$ and the equation for the spherical circle

$$\left.\begin{aligned} r_\nu &= a_\nu = \varkappa_\nu\ 90° \\ \nu\vartheta_\nu &= 90° - \varphi_\nu \\ \alpha_\nu &= \alpha_1 = e = 0 \\ \nu &= \sin a_\nu \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (135)$$

b) The combination of rolling curves to the real spherical conic section to form rolling curve pairs and the angles between the centres of rotation involved

From the derivations of the equations of the rolling curves, it can be seen (cf. pages 45 and 46) that all curves belonging to the same rolling curve family, i.e. having the same a_1- and α_1-values, can form rolling curve pairs with the basic curve (the real spherical conic

section), if both the basic curve and its rolling curve are either main curves or antipode curves. Furthermore, by altering the placing of the curves we can obtain two different angles between the centres of rotation,
a smaller angle determined by $C_\nu = a_1 (1 - \varkappa_\nu)$, for which $\varphi_\nu = \varphi_1$
a larger angle determined by $C_\nu = a_1 (1 + \varkappa_\nu)$, for which $\varphi_\nu = -\varphi_1$

If we imagine two rolling curves rolling simultaneously on the same basic curve in such a way that the point of contact is common to all three curves and corresponds to the value φ_1, we then find that two rolling curves can also form rolling curve pairs with each other for two different angles between the centres of rotation, namely

$$\left.\begin{aligned} &\text{a smaller } C_{L-R} = a_{\nu L} - a_{\nu R} = a_1 (\varkappa_L - \varkappa_R) \text{ for which } \varphi_R = \varphi_L \\ &\text{a larger } C_{L-R} = a_{\nu L} + a_{\nu R} = a_1 (\varkappa_L + \varkappa_R) \text{ for which } \varphi_R = -\varphi_L \end{aligned}\right\} \quad \ldots\ldots \quad (136)$$

where the suffixes L and R indicate the L- and R-rolling curves.

For spherical rolling curves, however, it is possible to replace the assumed centre of rotation for any of the rolling curves by its antipode. For rolling curves to the real spherical conic section, which are symmetrical with respect to the polar axis, this is, however, the same as replacing one of the curves by the antipode curve. At the same time, however, the φ_ν-value valid for the main curve must be replaced by the value $180° - \varphi_\nu$ for the antipode curve, which is obvious from the fact that φ for the point of contact moves in the opposite direction on the antipode curve, and that for a point of contact corresponding to $\varphi_\nu = 90°$ for the main curve we must have the same value of φ_ν for the antipode curve.

We always assume that

$$-180° < r < +180° \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (137)$$

for both the main curves and the antipode curves.

Furthermore, because $\tan (r_\nu - a_\nu)$ for the same value of φ_ν is the same for both a main curve and its antipode curve, we have

$$\left.\begin{aligned} r_{\nu A} &= r_{\nu M} - 180° \text{ if } r_{\nu M} \text{ is positive} \\ r_{\nu A} &= r_{\nu M} + 180° \text{ if } r_{\nu M} \text{ is negative} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (138)$$

where suffix M denotes the main curve and suffix A the antipode curve.

If a main curve is replaced by its antipode curve, it is now easy to understand that we obtain the following laws for the angle C_{L-R} between the centres of rotation, if the positive direction of C_{L-R} is measured in the same way as explained on page 5, and is assumed to lie on that part of the great circle which is covered by one or both radius vectors.

If the original curve has a positive value of r_ν, then $a_\nu = \varkappa_\nu a_1$ for the original curve is replaced by $a_\nu - 180°$ in eq. (136).

If the original curve has a negative value of r_ν, then $a_\nu = \varkappa_\nu a_1$ for the original curve is replaced by $a_\nu + 180° = \varkappa_\nu a_1 + 180°$ in eq. (136).

By repeated application of these laws, both main curves may be replaced by their antipode curves.

By replacing any of the curves, its nature of meshing is changed at the same time from external (or double internal) to internal meshing or vice versa.

The case where we obtain $|C_{L-R}| > 180°$ corresponds to double internal meshing. It is, however, convenient to measure the angle between the centres of rotation the shortest way along the great circle. If, therefore, the angle between the centres of rotation obtained from the above laws becomes positive and $> 180°$, we use instead the angle C_{L-R} = the angle determined as above $- 360°$, and if the angle becomes negative and its absolute value is $> 180°$, we use an angle C_{L-R} = the angle determined as above $+ 360°$. For the different combinations of rolling curves to the real spherical conic section we then obtain the following possible combinations:

Rolling case 1

where both curves are main curves or both antipode curves

$$\left.\begin{aligned} &\varphi_R = \varphi_L \\ &C_{L-R} = a_{rL} - a_{rR} = a_1\ (\varkappa_L - \varkappa_R) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (139)$$

Rolling case 2

where both curves are main curves or both antipode curves

$$\left.\begin{aligned} &\varphi_R = -\varphi_L \\ &C_{L-R} = a_{rL} + a_{rR} = a_1\ (\varkappa_L + \varkappa_R) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (140)$$

Rolling case 3

where one curve is a main curve and the other one an antipode curve

$$\left.\begin{aligned} &\varphi_R = 180° - \varphi_L \\ &C_{L-R} = a_{rL} - a_{rR} \mp 180° = a_1\ (\varkappa_L - \varkappa_R) \mp 180° \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (141)$$

where the upper sign is valid if $r_L > r_R$, and vice versa.

Rolling case 4

where one of the curves is a main curve and the other one an antipode curve

$$\left.\begin{aligned} &\varphi_R = 180° + \varphi_L \\ &C_{L-R} = a_{rL} + a_{rR} - 180° = a_1\ (\varkappa_L + \varkappa_R) - 180° \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (142)$$

We have $0 < \varkappa_r a_1 < 90°$. Thus, if one of the curves, e.g. the L-curve, is given, then it can be seen from the last four systems of equations that, for any given positive or negative value of C_{L-R} (where $|C_{L-R}| \leq 180°$), always one, but one only, of the four systems of equations gives a solution. The expression $C_{L-R} - \varkappa_L a_1$ can namely vary within the range $- 270°$ to $+ 180°$ and accordingly

eqs. (139) give a solution if $-90° < C_{L-R} - \varkappa_L a_1 < 0$
" (140) " " " " $0 < C_{L-R} - \varkappa_L a_1 < 90°$
" (141) " " " " $-270° < C_{L-R} - \varkappa_L a_1 < -180°$ or
$90° < C_{L-R} - \varkappa_L a_1 < 180°$
" (142) " " " " $-180° < C_{L-R} - \varkappa_L a_1 < -90°$

Therefore, for a given value of C_{L-R}, we have only one mating curve to a given rolling curve.

For bevel gears we have always external meshing and the absolute value of the shaft angle $< 180°$. It may be of interest to have a compilation of the combinations of rolling curves which give a meshing of this nature. For this purpose we investigate the nature of meshing if a main rolling curve rolls on another main rolling curve from $\varphi = 0$ to $\varphi = 360°$. The nature of meshing can only be altered, and is always altered, when the radius vector of one of the rolling curves changes its sign, i.e. when a larger branch with imaginary ν-value

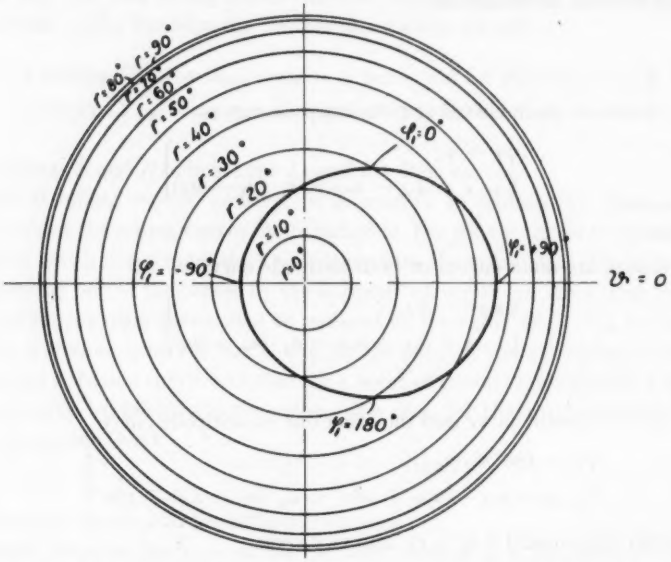


Fig. 6. The real spherical conic section (main curve) having $a_1 = 29.670°$ and $\alpha_1 = 30°$.

changes to a smaller branch, or vice versa. It is therefore easy to determine the nature of meshing during the rolling, if we observe that a curve which either has a real value of ν or is a larger branch with imaginary ν-value gives internal meshing for the distance $a_1(\varkappa_L - \varkappa_R)$ between the centres of rotation, and external meshing if this distance is $a_1(\varkappa_L + \varkappa_R)$ in the case where also the mating curve is any of these types of curve. If, however, the mating curve is a smaller branch with imaginary ν-value, we obtain the opposite nature of meshing. In the cases where we obtain internal meshing during the rolling, the internally meshed curve, i.e. the curve having the larger r-value which also has the larger $\varkappa$-value, is replaced by its antipode curve. Then the relation between φ_R and φ_L and the distance C_{L-R} between the centres of rotation are also altered according to the laws given above. We then obtain external meshing. In this way we can obtain all the possible combinations giving external meshing and having $|C_{L-R}| < 180°$, which are tabulated below.

Fig. 6 shows a real conic section curve having $a_1 = 29.670°$ and $\alpha_1 = 30°$, Fig. 7a a rolling curve to it with periodic number $\nu = 2$, and Fig. 7b a rolling curve with imaginary periodic number $|\nu| = {}^1/_2$.

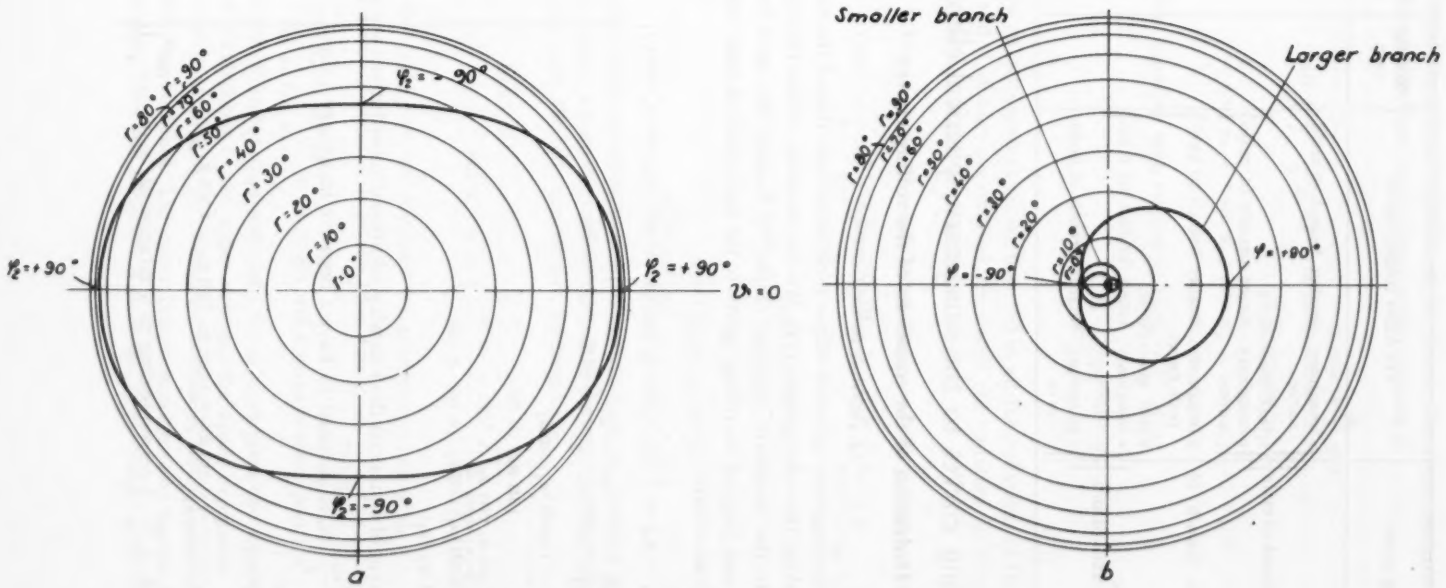


Fig. 7. Rolling curves (main curves) to the real spherical conic section in Fig. 6.
a) the two-period rolling curve
b) the smaller and larger branches of the rolling curve with imaginary periodic number $|\nu| = \frac{1}{2}$.

The one rolling curve	The other rolling curve	Rolling case	At point of contact $\varphi_R =$
ν real, main curve ν imaginary, larger branch of main curve	ν imaginary, smaller branch of main curve	1	$+\varphi_L$
	ν real, main curve ν imaginary, larger branch of main curve	2	$-\varphi_L$
ν imaginary, smaller branch of main curve	ν imaginary, smaller branch of antipode curve	3	$180^\circ - \varphi_L$
ν real, antipode curve ν imaginary, larger branch of antipode curve	ν real, main curve ν imaginary, larger branch of main curve		
	ν imaginary, smaller branch of main curve	4	$180^\circ + \varphi_L$

B. Rolling curves to the imaginary spherical ellipse

a) Deduction of the equations of the rolling curves

1) *Normal rolling curves*

The equation of the imaginary spherical ellipse is obtained from that of the real spherical conic section by making the substitutions (71). We can therefore obtain the equations for the rolling curves to the imaginary spherical ellipse, by making the same substitutions in eqs. (107), (108) and (109) of the rolling curves to the real spherical conic section. From eqs. (107) and (108) we obtain

$$\left.\begin{aligned}
&\tan(r_\nu - a_\nu) = \frac{1}{i}\tan a_\nu \sin\alpha_\nu \tan\varphi_\nu = \tanh|a_1|\sin\alpha_1\tan\varphi_\nu\\
&\tan\nu\frac{\vartheta_\nu}{2} = \tan\frac{90^\circ-\alpha_\nu}{2}(\cos\varphi_\nu + i\sin\varphi_\nu)\\
&\sin\alpha_\nu = \frac{i\tanh|a_1|\sin\alpha_1}{\tan a_\nu}\\
&\nu = \frac{\sin a_\nu\cos\alpha_\nu}{i\sinh|a_1|\cos\alpha_1}\\
&a_\nu = i\varkappa_\nu|a_1|
\end{aligned}\right\}$$

In the above system of equations the following additional substitutions are made

$$\left.\begin{aligned}
&\varkappa_\nu = \frac{|\varkappa_\nu|}{i}\ (\varkappa_\nu \text{ is assumed to have a negative imaginary value})\\
&\nu = \frac{|\nu|}{i}\\
&\cos\alpha_\nu = \frac{1}{\cos\bar\alpha_\nu},\ \sin\alpha_\nu = i\tan\bar\alpha_\nu,\ \tan\alpha_\nu = i\sin\bar\alpha_\nu\\
&\vartheta_\nu = \bar\vartheta_\nu \pm \frac{90^\circ}{\nu}\left(\text{corresponding to a rotation of } + \text{ or } -\frac{90^\circ}{\nu}\right)
\end{aligned}\right\} \quad \ldots\ldots\ (143)$$

We first treat the second equation in the first of the above systems of equations. After inserting the substitutions (143), the equation reduces to

$$\left.\begin{aligned} \tan\left(\frac{|\nu|}{i}\cdot\frac{\bar{\vartheta}_\nu}{2}\pm 45^\circ\right) &= \frac{-i\tanh|\nu|\frac{\bar{\vartheta}_\nu}{2}\pm 1}{1\pm i\tanh|\nu|\frac{\bar{\vartheta}_\nu}{2}} = \\ &= (\cos\bar{\alpha}_\nu - i\sin\bar{\alpha}_\nu)\,(\cos\varphi_\nu + i\sin\varphi_\nu) = e^{-i(\bar{\alpha}_\nu-\varphi_\nu)} \end{aligned}\right\}$$

Solving this equation with respect to $\tanh|\nu|\frac{\bar{\vartheta}_\nu}{2}$ gives

$$\tanh|\nu|\frac{\bar{\vartheta}_\nu}{2} = \frac{\tan}{\cot}\frac{\bar{\alpha}_\nu-\varphi_\nu}{2}$$

If now the substitutions (143) are inserted in all the equations of the original system of equations, and if all bars are omitted, we obtain the equation for the rolling curve to the imaginary spherical ellipse in parametric form

$$\left.\begin{aligned} \tan(r_\nu - a_\nu) &= \tan a_\nu\tan\alpha_\nu\tan\varphi_\nu = \tanh|a_1|\sin\alpha_1\tan\varphi_\nu \\ \tanh|\nu|\frac{\vartheta_\nu}{2} &= \frac{\tan}{\cot}\frac{\alpha_\nu-\varphi_\nu}{2} \text{ or } \operatorname{amh}|\nu|\,\vartheta_\nu = \frac{\alpha_\nu-\varphi_\nu}{180^\circ-(\alpha_\nu-\varphi_\nu)} \end{aligned}\right\} \quad \ldots\ldots\ (144)$$

where

$$\left.\begin{aligned} \tan\alpha_\nu &= \frac{\tanh|a_1|\sin\alpha_1}{\tan a_\nu} \\ |\nu| &= \frac{\sin a_\nu}{\sinh|a_1|\cos\alpha_1\cos\alpha_\nu} \\ a_\nu &= |\varkappa_\nu|\,|a_1| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (145)$$

The corresponding equation in non-parametric form is obtained by making the substitutions (71) and (143) in eq. (109) for the rolling curve to the real spherical conic section. If, at the same time, the bars are left out, we then have

$$\tan r_\nu = \frac{\sin a_\nu\cos a_\nu}{\sin\alpha_\nu\cos\alpha_\nu\left(\frac{\cos^2\alpha_\nu-\sin^2 a_\nu}{\sin\alpha_\nu\cos\alpha_\nu}\pm\sinh|\nu|\,\vartheta_\nu\right)} \quad \ldots\ldots\ (146)$$

Eqs. (144) and (146) give real solutions for arbitrary values of a_ν.

From the first equation in (145) we have $\tan\alpha_\nu < \frac{1}{\tan a_\nu}$ or $\alpha_\nu < 90^\circ - a_\nu$. Thus the expression $\frac{\cos^2\alpha_\nu-\sin^2 a_\nu}{\sin\alpha_\nu\cos\alpha_\nu}$ in eq. (146) will always become positive.

As may be seen from eqs. (144) and (146), the solution gives two separate pairs of branches. The one pair corresponds to the upper sign and the upper expression. They are obtained only if $\alpha_\nu - 90^\circ < \varphi_\nu < \alpha_\nu + 90^\circ$, because otherwise ϑ_ν would be complex. These branches are called the tan-branches. The other branches correspond to the lower sign and the lower expression. They are obtained if $\alpha_\nu + 90^\circ < \varphi_\nu < \alpha_\nu + 270^\circ$. These branches are

called the cot-branches. Similarly as for the imaginary spherical ellipse, the main curve is defined as the curve obtained if we choose r_ν—a_ν in the same quadrant as φ_ν. From eqs. (144) we can draw the conclusion that the above condition is identical with the following:

For the tan-branch of the main curve and the cot-branch of the antipode curve, r_ν lies in the first or second quadrant. For the cot-branch of the main curve and the tan-branch of the antipode curve, r_ν lies in the third or fourth quadrant.

Fig. 8 shows a rolling curve to the imaginary spherical ellipse having $\alpha_1 = 30°$, $a_\nu = 30°$ and $|\nu| = 1$.

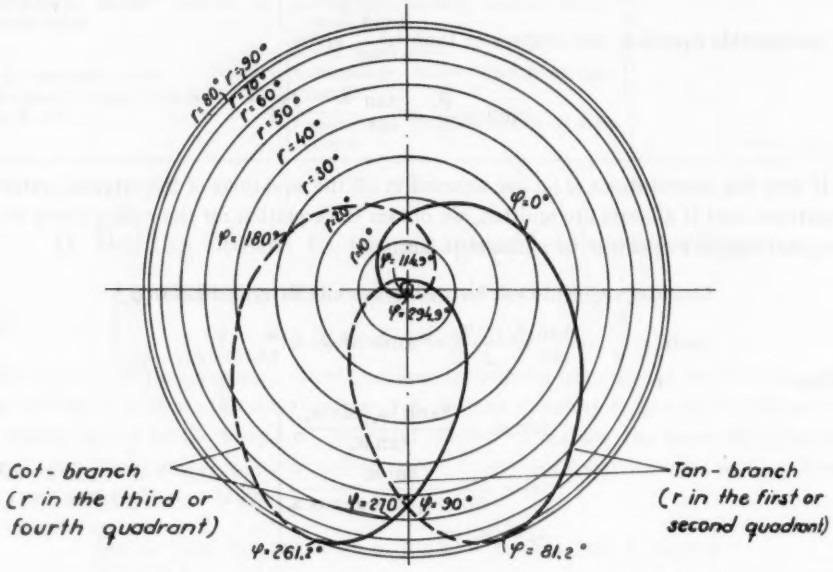


Fig. 8. Rolling curve (main curve) to the imaginary spherical ellipse having $a_\nu = |\varkappa_\nu| \cdot |a_1| = 30°$, $\alpha_1 = 30°$ and $|\nu| = 1$.

Eq. (146) may be transformed into the same form as eqs. (112) and (118). Since the upper sign is valid for the tan-branch, in which case k_1 and k_2 are negative, and the lower sign applies to the cot-branch, in which case k_1 an k_2 are positive, we first obtain

$$\left.\begin{aligned}
\tan r_\nu &= \frac{k_1}{k_2 - \sinh|\nu|\,\vartheta_\nu} = \frac{\mp|k_1|}{\mp|k_2| - \sinh|\nu|\,\vartheta_\nu} \\
|k_1| &= \frac{\sin a_\nu \cos a_\nu}{\sin \alpha_\nu \cos \alpha_\nu} = \frac{\sin 2a_\nu}{\sin 2\alpha_\nu} \\
|k_2| &= \frac{\cos^2 \alpha_\nu - \sin^2 a_\nu}{\sin \alpha_\nu \cos \alpha_\nu} = \frac{\cos 2\,a_\nu + \cos 2\alpha_\nu}{\sin 2\,\alpha_\nu} \\
\tan \alpha_\nu &= \frac{\tanh|a_1| \sin \alpha_1}{\tan a_\nu} \\
|\nu| &= \frac{\sin a_\nu}{\sinh|a_1| \cos \alpha_1 \cos \alpha_\nu}
\end{aligned}\right\}$$

From the second and third equations, we then form expressions for $\sin 2\,a_\nu$ and $\cos 2a_\nu$, from which expressions we can solve with respect to $\tan 2\,a_\nu$ and $\tan 2\,\alpha_\nu$. Since k_1 and k_2 have the same signs, we obtain

$$\tan 2\,\alpha_\nu = \frac{2\,|k_2|}{k_1^2 + k_2^2 - 1}$$

$$\tan 2\,a_\nu = \frac{2\,k_1\,k_2}{k_2^2 - k_1^2 + 1}$$

From the expressions for $\tan \alpha_\nu$ and $|\nu|$ we can solve for $\sinh |a_1|$ and $\cos \alpha_1$, obtaining

$$\sinh |a_1| = \frac{\sin a_\nu}{|\nu| \cos \alpha_1 \cos \alpha_\nu}$$

$$\cos \alpha_1 = \frac{1}{\cos \alpha_\nu} \sqrt{\frac{\cos^2 a_\nu - \sin^2 \alpha_\nu}{\cos^2 a_\nu + |\nu|^2 \sin^2 \alpha_\nu}}$$

The original system of equations now takes the form

$$\left.\begin{aligned}
&\tan r_\nu = \frac{k_1}{k_2 - \sinh |\nu|\,\vartheta_\nu} = \frac{\mp |k_1|}{\mp |k_2| - \sinh |\nu|\,\vartheta_\nu} \\
&\tan 2\,a_\nu = \frac{2\,k_1\,k_2}{k_2^2 - k_1^2 + 1} \\
&\tan 2\,\alpha_\nu = \frac{2\,|k_\nu|}{k_1^2 + k_2^2 - 1} \\
&\cos \alpha_1 = \frac{1}{\cos \alpha_\nu} \sqrt{\frac{\cos^2 a_\nu - \sin^2 \alpha_\nu}{\cos^2 a_\nu + |\nu|^2 \sin^2 \alpha_\nu}} \\
&\sinh |a_1| = \frac{\sin a_\nu}{|\nu| \cos \alpha_1 \cos \alpha_\nu} \\
&(\tanh |a_1| \sin \alpha_1 = \tan a_\nu \tan \alpha_\nu = \tanh e\,|a_1|)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (147)$$

By comparing eqs. (146) and (147), it can be seen that in the above system of equations we must have either both k_1 and k_2 negative, which corresponds to a tan-branch and the upper signs in the first equation, or both k_1 and k_2 positive, which corresponds to a cot-branch and the lower signs in the first equation. We always choose α_ν, α_1 and a_ν in the first quadrant.

This system of equations is used if we have the equation of the curve given in the same form as the first equation in the system. By using the following equations it is then possible to determine α_ν, $a_\nu = |\varkappa_\nu||a_1|$, α_1 and a_1, after which the equation may be written in any of the forms (144) and (146).

The condition for obtaining real solutions from (147) is that $\cos \alpha_1 < 1$, i.e.

$$\cos^2 a_\nu - \sin^2 \alpha_\nu \geq 0$$

and

$$\cos^2 a_\nu - \sin^2 \alpha_\nu \leq \cos^2 a_\nu \cos^2 \alpha_\nu + |\nu|^2 \sin^2 \alpha_\nu \cos^2 \alpha_\nu$$

The last condition may be transformed into

$$\cos^2 a_\nu \leq 1 + |\nu|^2 \cos^2 \alpha_\nu$$

which is always fulfilled.

Since α_ν and a_ν both lie in the first quadrant, the first condition can also be written in the form

$$\cos a_\nu \geq \sin \alpha_\nu, \text{ from which } 90^\circ - a_\nu \geq \alpha_\nu$$

or

$$2\,a_\nu + 2\,\alpha_\nu \leq 180^\circ$$

Three different cases are obtained. Firstly, the angles $2\,\alpha_\nu$ and $2\,a_\nu$ may both lie in the first quadrant. The condition is then always fulfilled. Secondly, these angles may both lie in the second quadrant. If $2\,\alpha_\nu$ lies in the second quadrant, we then have, according to the third equation in (147), $k_2^2 + k_1^2 < 1$ or $1 - k_1^2 > k_2^2$, i.e. $1 - k_1^2$ is positive. According to the second equation in (147), $\tan 2\,a_\nu$ is in this case positive, i.e. the angle $2\,a_\nu$ lies in the first quadrant. The two angles $2\,\alpha_\nu$ and $2\,a_\nu$ cannot therefore both lie in the second quadrant. The case now only remains where one of the angles lies in the first and the other in the second quadrant. Irrespective of which of the angles lies in the first or second quadrant, the above condition is then equivalent to

$$\tan 2\,\alpha_\nu \leq -\tan 2\,a_\nu$$

or, together with the second and third equations in (147),

$$\frac{2\,|k_2|}{k_1^2 + k_2^2 - 1} \leq -\frac{2\,|k_1|\,|k_2|}{k_2^2 - k_1^2 + 1}$$

Since the angles lie in different quadrants, the denominators have different signs. The condition can thus be transformed into

$$k_1^2 - k_2^2 - 1 \leq |k_1|\,(k_1^2 + k_2^2 - 1)$$

or

$$[(|k_1| - 1)^2 + k_2^2]\;(|k_1| + 1) \geq 0$$

which condition is always fulfilled. The system of equations (147), therefore, will always give real solutions.

2) Special cases of rolling curves to the imaginary spherical ellipse

α) The crown-wheel curve (the spherical sinh-curve having $|\tan r_\nu \sinh |\nu|\,\vartheta_\nu| > 1$*)*

From eq. (144) it can be seen that the mean radius of the rolling curve equals a_ν. The crown-wheel curve is therefore obtained if we choose $a_\nu = 90^\circ$. Substituting this value in eqs. (145) we obtain

$$\left.\begin{aligned} \tan \alpha_\nu &= \frac{\tanh |a_1| \sin \alpha_1}{\tan 90^\circ} \\ |\nu| &= \frac{1}{\sinh |a_1| \cos \alpha_1 \cos \alpha_\nu} \end{aligned}\right\}$$

from which

$$\left.\begin{aligned} \alpha_\nu &= 0 \\ |\nu| &= \frac{1}{\sinh |a_1| \cos \alpha_1} \end{aligned}\right\}$$

The system of equations (144) now takes the form

$$\left.\begin{aligned} &\tan(r_\nu - 90^\circ) = \tanh|a_1| \sin\alpha_1 \tan\varphi_\nu \\ &\tanh|\nu|\frac{\vartheta_\nu}{2} = \frac{\tan}{\cot}\left(-\frac{\varphi_\nu}{2}\right) \text{ or } \operatorname{amh}|\nu|\,\vartheta_\nu = \frac{-\varphi_\nu}{180^\circ + \varphi_\nu} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (148)$$

where

$$\left.\begin{aligned} &\alpha_\nu = 0 \\ &|\nu| = \frac{1}{\sinh|a_1| \cos\alpha_1} \\ &a_\nu = |\varkappa_\nu|\,|a_1| = 90^\circ \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (149)$$

which is the equation of the crown-wheel curve in parametric form.

The equation in non-parametric form is obtained by substituting the above values of a_ν and α_ν in eq. (146) and using the relation $\tan a_\nu \tan\alpha_\nu = \tanh|a_1| \sin\alpha_1$. Thus

$$\tan r_\nu = \pm\frac{1}{\tanh|a_1| \sin\alpha_1 \sinh|\nu|\,\vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots \quad (150)$$

In eqs. (148) and (150) the upper expressions and the upper signs correspond to the tan-branch, and the lower expressions and the lower signs correspond to the cot-branch.

The crown-wheel curve now treated is included in eqs. (147), in which case $k_1 > 1$, $\alpha_\nu = 0$ and $k_2 \to \mp 0$, i.e. $|k_2| \to +0$. We then obtain from eqs. (147)

$$\tan 2a_\nu \to -0, \text{ thus } a_\nu = |\varkappa_\nu|\,|a_1| = 90^\circ$$
$$\tan 2\alpha_\nu = +0, \text{ thus } \alpha_\nu = 0$$

from which

$$\sinh|a_1| = \frac{1}{|\nu| \cos\alpha_1}$$

and by comparing eq. (150)

$$|k_1| = \frac{1}{\tanh|a_1| \sin\alpha_1}$$

The expression for $\cos\alpha_1$ in eqs. (147) becomes indeterminate, but its limiting value can be obtained from the last two equations. If we solve with respect to $\cos\alpha_1$ by eliminating $|a_1|$, we obtain

$$\cos\alpha_1 = \sqrt{\frac{k_1^2 - 1}{k_1^2 + |\nu|^2}}$$

For the crown-wheel curve the system of eqs. (147) thus takes the form

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\sinh|\nu|\,\vartheta_\nu} = \frac{\mp|k_1|}{-\sinh|\nu|\,\vartheta_\nu} \\ &a_\nu = 90^\circ \\ &\alpha_\nu = 0 \\ &\cos\alpha_1 = \sqrt{\frac{k_1^2 - 1}{k_1^2 + |\nu|^2}} \\ &\sinh|a_1| = \frac{1}{|\nu| \cos\alpha_1} = \sqrt{\frac{k_1^2 + |\nu|^2}{|\nu|^2 (k_1^2 - 1)}} \\ &(\tanh|a_1| \sin\alpha_1 = \tanh e\,|a_1| = \frac{1}{|k_1|} < 1) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (151)$$

If radii to the curve are projected from the centre of the sphere on to a cylinder tangential to the great circle $r = 90°$, and if we place the x-axis on the great circle and the y-axis along a generatrix of the cylinder, the equation of the projected curve takes the form

$$\frac{y}{\tanh |a_1| \sin \alpha_1} = \pm \sinh |\nu| \vartheta_\nu = \pm \sinh \frac{x}{\sinh |a_1| \cos \alpha_1}$$

For this reason, therefore, the curve is called the spherical sinh-curve.

The condition for obtaining real solutions from eqs. (151) is that $\cos \alpha_1$ becomes real and ≤ 1 and $\sinh |a_1|$ finite, i.e.

$$|k_1| > 1$$

or, using eq. (150) and the last equation in (151),

$$|\tan r_\nu \sinh |\nu| \vartheta_\nu| = |k_1| > 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (152)$$

Eqs. (148) and (150) thus do not include all spherical sinh-curves, but only those which satisfy condition (152). In the two following sections it will be shown that the remaining spherical sinh-curves are obtained as other special cases of rolling curves to the imaginary spherical ellipse.

β) The rolling curve with $|\varkappa_\nu| |a_1| = a_\nu = 0$ *(the spherical sinh-curve having* $|\tan r_\nu \sinh |\nu| \vartheta_\nu| < 1$*)*

If we put $a_\nu = |\varkappa_\nu| |a_1| = 0$, we obtain from eqs. (145)

$$\left.\begin{aligned} &\tan \alpha_\nu = \frac{\tanh |a_1| \sin \alpha_1}{\tan a_\nu} = \infty, \text{ i.e. } \alpha_\nu = 90° \\ &|\nu| = \frac{\sin a_\nu}{\sinh |a_1| \cos \alpha_1 \cos \alpha_\nu} = \frac{\cos a_\nu \tan \alpha_1}{\cosh |a_1| \sin \alpha_\nu} = \frac{\tan \alpha_1}{\cosh |a_1|} \end{aligned}\right\}$$

The systems of equations (144) and (145) now take the form

$$\left.\begin{aligned} &\tan r_\nu = \tanh |a_1| \sin \alpha_1 \tan \varphi_\nu \\ &\tanh |\nu| \frac{\vartheta_\nu}{2} = \frac{\tan}{\cot} \frac{90° - \varphi_\nu}{2} \text{ or amh } |\nu| \vartheta_\nu = \frac{90° - \varphi_\nu}{90° + \varphi_\nu} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (153)$$

where

$$\left.\begin{aligned} &\alpha_\nu = 90° \\ &|\nu| = \frac{\tan \alpha_1}{\cosh |a_1|} \\ &a_\nu = |\varkappa_\nu| |a_1| = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (154)$$

which is thus the equation in parametric form of the rolling curve having $a_\nu = |\varkappa_\nu| |a_1| = 0$.

The equation in non-parametric form can be obtained by inserting the above values of α_ν and $|\nu|$ in eq. (146), giving

$$\tan r_\nu = \pm \frac{\tanh |a_1| \sin \alpha_1}{\sinh |\nu| \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots \quad (155)$$

In eqs. (153) and (155) the upper signs and the upper expressions correspond to the tan-branch. The lower signs and the lower expressions correspond to the cot-branch.

The curve investigated above is included in eqs. (147), in which case $k_2 \to \mp 0$, i.e. $|k_2| \to 0$ and $\alpha_\nu = 90°$. We then obtain from eqs. (147)

$$\tan 2\, a_\nu = 0, \text{ thus } a_\nu = |\varkappa_\nu|\,|a_1| = 0$$
$$\tan 2\, \alpha_\nu = -0, \text{ thus } \alpha_\nu = 90°$$

By using the sixth equation in (147), the expression for $\sinh |a_1|$ reduces to

$$\cosh |a_1| = \frac{\cos a_\nu \tan \alpha_1}{|\nu| \sin \alpha_\nu} = \frac{\tan \alpha_1}{|\nu|}$$

By comparing eqs. (147) and (155) we find further that

$$|k_1| = \tanh |a_1| \sin \alpha_1$$

The fourth equation in (147) gives an indeterminate value of $\cos \alpha_1$, but by eliminating $|a_1|$ from the last two expressions for $\cosh |a_1|$ and $|k_1|$ we obtain

$$\cos \alpha_1 = \sqrt{\frac{1 - k_1^2}{1 + |\nu|^2}}$$

The system of equations (147) now takes the form

$$\left.\begin{aligned}
&\tan r_\nu = \frac{k_1}{-\sinh |\nu|\, \vartheta_\nu} = \frac{\mp |k_1|}{-\sinh |\nu|\, \vartheta_\nu} \\
&a_\nu = 0 \\
&\alpha_\nu = 90° \\
&\cos \alpha_1 = \sqrt{\frac{1 - k_1^2}{1 + |\nu|^2}} \text{ or } \tan \alpha_1 = \sqrt{\frac{|\nu|^2 + k_1^2}{1 - k_1^2}} \\
&\cosh |a_1| = \frac{\tan \alpha_1}{|\nu|} = \sqrt{\frac{k_1^2 + |\nu|^2}{|\nu|^2 (1 - k_1^2)}} \\
&(\tanh |a_1| \sin \alpha_1 = |k_1| < 1)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (156)$$

If radii to the curve are projected from the centre of the sphere on to a cylinder tangential to the great circle $r = 90°$, and if the x-axis is located on the great circle and the y-axis along a generatrix of the cylinder, the equation of the projected curve becomes

$$y \tanh |a_1| \sin \alpha_1 = \pm \sinh |\nu|\, \vartheta_\nu = \pm \sinh \left(\frac{\tan \alpha_1}{\cosh |a_1|}\, x \right)$$

On the basis of the last equation, this curve, like the curve on page 70, is called the spherical sinh-curve.

The condition for obtaining real solutions from eqs. (156) is that $\cos \alpha_1$ becomes real and ≤ 1, and $\cosh |a_1|$ finite, i.e.

$$|k_1| < 1$$

or, using eq. (155) and the last equation in (156),

$$|\tan r_\nu \sinh |\nu|\, \vartheta_\nu| = |k_1| < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (157)$$

Eqs. (153) and (155) therefore only include sinh-curves which satisfy eq. (157).

γ) The rolling curve with $\alpha_1 = 90°$ *(the spherical sinh-curve having* $|\tan r_\nu \sinh |\nu| \vartheta_\nu| = 1$*)*

By inserting the first equation of (145) in the second equation, we obtain

$$|\nu| = \frac{\sin a_\nu}{\sinh |a_1| \cos \alpha_1 \cos \alpha_\nu} = \frac{\tan \alpha_1 \cos a_\nu}{\cosh |a_1| \sin \alpha_\nu}$$

Since $\alpha_1 = 90°$, we must thus have $|a_1| = \infty$, because otherwise $|\nu|$ would become infinite.

From the first equation in (145) we now obtain

$$\tan \alpha_\nu = \frac{1}{\tan a_\nu} \text{ or } \alpha_\nu = 90° - a_\nu$$

Eqs. (144) and (145) now reduce to

$$\left.\begin{aligned} &\tan (r_\nu - a_\nu) = \tan \varphi_\nu \\ &\tanh |\nu| \frac{\vartheta_\nu}{2} = \begin{matrix}\tan \\ \cot\end{matrix} \frac{\alpha_\nu - \varphi_\nu}{2} \text{ or amh } |\nu| \vartheta_\nu = \begin{matrix}\alpha_\nu - \varphi_\nu \\ 180° - (\alpha_\nu - \varphi_\nu)\end{matrix} \end{aligned}\right\} \quad \ldots\ldots \text{(158)}$$

where

$$\left.\begin{aligned} &\alpha_\nu = 90° - a_\nu \\ &|\nu| = \lim \frac{\tan \alpha_1}{\cosh |a_1|} = \pm \tan \Phi \\ &a_\nu = |\varkappa_\nu| \, |a_1| = \text{arbitrary value} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(159)}$$

In eqs. (158) and (159), a_ν and $|\nu| = \pm \tan \Phi$ may be chosen arbitrarily, but $|\nu|$ must have the same value for both curves in a rolling curve pair.

The equation in non-parametric form can be obtained from eq. (146) by inserting $\alpha_\nu = 90° - a_\nu$. We then obtain

$$\tan r_\nu = \pm \frac{1}{\sinh |\nu| \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(160)}$$

In eqs. (158) and (160) the upper expressions and the upper signs correspond to the tan-branch, and the lover expressions and the lower signs to the cot-branch.

Eq. (160) can be transformed in the following way

$$\tan r_\nu = \frac{2 \tan \frac{r_\nu}{2}}{1 - \tan^2 \frac{r_\nu}{2}} = \pm \frac{2}{e^{|\nu| \vartheta_\nu} - e^{-|\nu| \vartheta_\nu}}$$

Solving with respect to $\tan \frac{r_\nu}{2}$ from the last two expressions, we obtain, observing that $(e^{|\nu| \vartheta_\nu} - e^{-|\nu| \vartheta_\nu})^2 + 4 = (e^{|\nu| \vartheta_\nu} + e^{-|\nu| \vartheta_\nu})^2$,

for the tan-branches	for the cot-branches
$\tan \frac{r_\nu}{2} = \frac{+ e^{-|\nu| \vartheta_\nu}}{- e^{+|\nu| \vartheta_\nu}}$	$\tan \frac{r_\nu}{2} = \frac{- e^{-|\nu| \vartheta_\nu}}{+ e^{+|\nu| \vartheta_\nu}}$

or

$$\tan \frac{r_\nu}{2} = \pm e^{\mp |\nu| \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(161)}$$

where $\bar{\vartheta}_\nu$ is measured in radians, and where
the upper sign before e gives the tan-branch of the main curve or the cot-branch of the antipode curve
the lower sign before e gives the cot-branch of the main curve or the tan-branch of the antipode curve
the upper sign before $|\nu|\,\bar{\vartheta}_\nu$ gives the main curve
the lower sign before $|\nu|\,\bar{\vartheta}_\nu$ gives the antipode curve.

These rules for the signs are easily determined from the fact (see p. 66) that for the main curves r_ν lies in the first or second quadrant for the tan-branch, and in the third or fourth quadrant for the cot-branch.

We now rotate the polar axis by putting

$$\vartheta = \bar{\vartheta} \mp \frac{1}{|\nu|} \ln \tan \frac{R_s}{2}$$

where $0 \leq R_s \leq 180°$ and the upper sign applies to the main curve and the lower sign to the antipode curve. The equation of the curves then takes the form

$$\left.\begin{aligned} &\tan \frac{r_\nu}{2} = \pm \left(\tan \frac{R_s}{2}\right) e^{-|\nu|\bar{\vartheta}_\nu} \text{ (for the main curve)} \\ \text{and} \\ &\tan \frac{180° - r_\nu}{2} = \mp \left(\tan \frac{180° - R_s}{2}\right) e^{-|\nu|\bar{\vartheta}_\nu} \text{ (for the antipode curve)} \end{aligned}\right\} \ldots (162)$$

where $\bar{\vartheta}_\nu$ is measured in radians, and where the upper signs correspond to the tan-branches and the lower signs to the cot-branches.

For $\bar{\vartheta}_\nu = 0$ we obtain $R_s = |r_\nu|$. Thus R_s is the initial value of the radius taken in such a direction that $0 \leq R_s \leq 180°$.

From the second equation in (159) we have the angle Φ from the radius vector to the normal given by $\tan \Phi = \pm |\nu|$. The angle Φ is thus constant. From this we can draw the conclusion that any two rolling curves having the same value of $|\nu|$ can form a rolling curve pair with an arbitrary angle between the axes of rotation, provided that both the curves are either main curves or antipode curves. The curve is in this respect analogous to the logarithmic spiral in plane geometry.

For the special case now treated, the equations (147) take the form

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\sinh |\nu|\,\vartheta_\nu} = \frac{\mp |k_1|}{-\sinh |\nu|\,\vartheta_\nu} \\ &a_\nu = \text{arbitrary value} \\ &\alpha_\nu = 90° - a_\nu \\ &\alpha_1 = 90° \\ &|a_1| = \infty \\ &(\tanh |a_1| \sin \alpha_1 = \tan a_\nu \tan \alpha_\nu = \tanh e\,|a_1| = |k_1| = 1) \end{aligned}\right\} \ldots\ldots (163)$$

As can be seen from eq. (160), this type of curve is also a spherical sinh-curve, i.e. the remaining sinh-curve for which, according to eqs. (160) and (163),

$$|\tan r_\nu \sinh |\nu|\,\vartheta_\nu| = |k_1| = 1 \ldots\ldots\ldots\ldots\ldots\ldots (164)$$

b) The combination of rolling curves to the imaginary spherical ellipse to form rolling curve pairs and the angles between the centres of rotation involved

The rolling curves to the imaginary spherical ellipse were obtained from the rolling curves to the real spherical conic section by means of the substitutions according to eqs. (71) and (143). Rolling curves to the real spherical conic section which have the same a_1- and α_1-values as the basic curve can form rolling curve pairs with it having the two angles between the centres of rotation $C_\nu = a_1 (1 \mp \varkappa_\nu)$, and with each other having the angles between the centres of rotation $C_{L-R} = a_1 (\varkappa_L \mp \varkappa_R)$, if the curves are both either main curves or antipode curves. Consequently, the rolling curves to the imaginary spherical ellipse with the same a_1- and α_1-values as the basic curve can also form rolling curves with the basic curve for two different angles between the centres of rotation $C_\nu = |a_1|(i \mp |\varkappa_\nu|)$, i.e. with complex angles between the centres of rotation, and with each other for the two real angles between the centres of rotation $C_{L-R} = |a_1|(|\varkappa_L| \mp |\varkappa_R|)$. Thus the rolling curves to the imaginary spherical ellipse are twin curves. For rolling curves to the real spherical conic section we have for the smaller angle between the centres of rotation $\varphi_R = \varphi_L$ and for the greater $\varphi_R = -\varphi_L$. The same two relations will then be valid also for rolling curves to the imaginary spherical ellipse, since these relations are not altered by the substitutions (71) and (143). For rolling curve pairs to the imaginary spherical ellipse we therefore obtain, if both curves are main curves or both curves antipode curves, in which case $|r_\nu - a_\nu|$ is equal for both curves,

$$\left.\begin{array}{l} C_{L-R} = |a_1|(|\varkappa_L| \mp |\varkappa_R|) \\ \text{in which } \varphi_R = +\varphi_L \text{ applies to the upper sign and} \\ \qquad\qquad \varphi_R = -\varphi_L \text{ to the lower sign.} \end{array}\right\} \ldots\ldots\ldots (165)$$

For spherical rolling curves, however, it is possible to change the nature of meshing from internal to external by replacing the original centre of rotation for the internally meshed rolling curve by its antipode. From eqs. (144) and Fig. 8 it can be seen that the tan- and cot-branches are congruent, because if for the tan-branch (or for the cot-branch) we replace φ_ν by $\pm 180° + \varphi_\nu$ and ϑ by $-\vartheta$, we obtain the cot-branch (or the tan-branch). Then also, according to the first equation in (144), $r_\nu - a_\nu$ changes to $r_\nu - a_\nu - 180°$, i.e. r_ν changes to $r_\nu - 180°$ if the original value of r_ν is positive, and to $r_\nu + 180°$ if the original r_ν-value is negative. These relations are the same as those given in eqs. (138). In the same way as for rolling curves to the real conic section, we presuppose that $-180° < r < +180°$ for both the main curves and the antipode curves. From this it can be understood that the replacement of the centre of rotation by its antipode, resulting in the sign for ϑ being changed, is the same as replacing the tan-branch by the cot-branch or vice versa, if at the same time φ_ν is replaced by $\varphi_\nu \pm 180°$ and C_{L-R} by $C_{L-R} \mp 180°$. For the different combinations of rolling curves to the imaginary spherical ellipse we thus obtain, by making such replacements, the following possibilities, if the angle between the centres of rotation is measured the shortest way along the great circle and in the same direction as the positive direction of r_ν.

Rolling case 1

$$\left.\begin{aligned} \varphi_R &= \varphi_L \\ C_{L-R} &= a_{rL} - a_{rR} = |a_1|\,(|\varkappa_L| - |\varkappa_R|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (166)$$

Rolling case 2

$$\left.\begin{aligned} \varphi_R &= -\varphi_L \\ C_{L-R} &= a_{rL} + a_{rR} = |a_1|\,(|\varkappa_L| + |\varkappa_R|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (167)$$

Rolling case 3

$$\left.\begin{aligned} \varphi_R &= 180^\circ - \varphi_L \\ C_{L-R} &= a_{rL} + a_{rR} - 180^\circ = |a_1|\,(|\varkappa_L| + |\varkappa_R|) - 180^\circ \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (168)$$

Rolling case 4

$$\left.\begin{aligned} \varphi_R &= 180^\circ + \varphi_L \\ C_{L-R} &= a_{rL} - a_{rR} \mp 180^\circ = |a_1|\,(|\varkappa_L| - |\varkappa_R|) \mp 180^\circ \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (169)$$

where the upper sign is valid if $a_{rL} > a_{rR}$ and vice versa.

For all four rolling cases the condition applies that both curves in the rolling curve pair must be either main curves or antipode curves. This is also evident from the fact that $\tan \Phi$ is positive for the main curves, but negative for the antipode curves.

In the same way as for rolling curves to the real conic section, we can find that, if a_r for one of the curves is given, the last four equations provide one, but only one, solution for which $a_r = |\varkappa_r||a_1| \leq 90^\circ$ for the other curve. Therefore, for a given value of C_{L-R}, we have one, but only one, mating curve to a given rolling curve.

To obtain a compilation of the combinations of rolling curves which give external meshing with the absolute value of the shaft angle $|C_{L-R}| < 180^\circ$, we can apply the same reasoning as was used for rolling curves to the real spherical conic section, i.e. we investigate the nature of meshing if a main rolling curve rolls on another main rolling curve. The nature of meshing can only be altered, and is always altered, when the radius vector of one of the rolling curves changes its sign, i.e. when a tan-branch changes to a cot-branch or vice versa. It is therefore easy to determine the nature of meshing during rolling, if we imagine the curves rolling on each other, and if we observe that two main curve tan-branches have external meshing for the larger, and internal meshing for the smaller, angle between the centres of rotation. In the cases where we obtain internal meshing during the rolling, the internally meshed curve is replaced by a cot-branch if it is a tan-branch and vice versa. At the same time, φ_r is also replaced by $\varphi_r \pm 180^\circ$, and the angle between the centres of rotation changes $\pm 180^\circ$. The other rolling curve, i.e. the curve which changed its sign, is, however, retained. In this way we can obtain all possible combinations giving external meshing and having $|C_{L-R}| < 180^\circ$, these being tabulated below

The rolling curve having the greatest a_ν-value	The rolling curve having the smallest a_ν-value	Rolling case	At point of contact $\varphi_R =$
tan-branch of main curve cot-branch of antipode curve	cot-branch of main curve tan-branch of antipode curve	1	φ_L
tan-branch of main curve cot-branch of antipode curve	tan-branch of main curve cot-branch of antipode curve	2	$-\varphi_L$
cot-branch of main curve tan-branch of antipode curve	cot-branch of main curve tan-branch of antipode curve	3	$180^\circ - \varphi_L$
cot-branch of main curve tan-branch of antipode curve	tan-branch of main curve cot-branch of antipode curve	4	$180^\circ + \varphi_L$

The above table was at first derived assuming that both curves in the rolling curve pair are main curves. If, however, the main curves are replaced by the corresponding antipode curves, these will also roll on each other with the same centres of rotation. The point of contact of the rolling curves will, however, now lie on the greater arc of the great circle through the centres of rotation, i.e. both curves are internally meshed. This nature of meshing may be reduced to normal external meshing by replacing the original centres of rotation by their antipodes. According to what is shown above, this is, however, the same as replacing both curves in such a way that a tan-branch is replaced by a cot-branch and vice versa. The relation between φ_R and φ_L remains unchanged as a result of these replacements. The above table also gives the rolling possibilities for antipode curves derived in this manner. In a rolling curve pair the two curves must be either both main curves or both antipode curves, which may also be seen from the fact that tan Φ, which must have the same value for both curves, has opposite signs for the main and antipode curves.

C. Rolling curves to the imaginary spherical hyperbola

a) Deduction of the equations of the rolling curves

1) Rolling curves with real periodic number

The equation of the imaginary spherical hyperbola was obtained from that of the real spherical conic section by means of the substitutions (86). It is, therefore, possible to obtain the equation of the rolling curve to the imaginary spherical hyperbola by making the same substitutions in the equations (102), (103) and (105) of the rolling curve to the real spherical conic section. If the suffix 1 indicates values belonging to the basic curve, we then have

$$\left.\begin{aligned}
&\tan\,(r_\nu - \varkappa_\nu\, i\,|a_1|) = \mp \tanh\,|a_1|\,\frac{1}{\tan\,\alpha_1}\cdot\frac{1}{\sin\,\varphi_1}\\
&\tan\,\nu\,\frac{\vartheta_\nu}{2} = \pm\, i\,\tan\frac{90^\circ - \alpha_\nu}{2}\tan\frac{90^\circ \mp \varphi_1}{2}\\
&\sin\,\alpha_\nu = \frac{\tanh\,|a_1|\,\tan\,(90^\circ + \varkappa_\nu\, i\,|a_1|)}{\tan\,\alpha_1}\\
&\nu = \frac{i\,\cos\,\alpha_\nu\,\cos\,(90^\circ + \varkappa_\nu\, i\,|a_1|)\,\sin\,\alpha_1}{\sinh\,|a_1|}
\end{aligned}\right\}$$

where the upper signs apply to the angle $a_1(1-\varkappa_\nu)$ and the lower signs to the angle $a_1(1+\varkappa_\nu)$ between the centres of rotation.

In these equations we now make the following further substitutions

$$\left.\begin{aligned}
&\sin\alpha_\nu = \frac{1}{\sin\bar{\alpha}_\nu},\ \cos\alpha_\nu = \frac{1}{i\tan\bar{\alpha}_\nu}\\
&\varkappa_\nu = i\,|\varkappa_\nu|\\
&\varkappa_\nu\, a_1 = -|\varkappa_\nu|\,|a_1| = a_\nu\\
&\bar{a}_\nu = 90^\circ + a_\nu = 90^\circ - |\varkappa_\nu|\,|a_1|\\
&\bar{a}_1 = 90^\circ + i\,|a_1|\\
&\varphi_\nu = \varphi_1, \text{ if the angle between the centres of rotation} = a_1(1-\varkappa_\nu) = \bar{a}_1 - \bar{a}_\nu\\
&\varphi_\nu = 180^\circ + \varphi_1, \text{ if the angle between the centres of rotation} =\\
&\qquad = a_1(1+\varkappa_\nu) = \bar{a}_1 + \bar{a}_\nu - 180^\circ\\
&\frac{\alpha_1}{2} = 90^\circ - \frac{\bar{\alpha}_1}{2}
\end{aligned}\right\} \quad (170)$$

The equation of the rolling curve will then become for both angles between the centres of rotation (if the bar in $\bar{\alpha}_\nu$ is omitted)

$$\left.\begin{aligned}
&\tan(r_\nu - \bar{a}_\nu) = \tan\bar{a}_\nu \sin\alpha_\nu \sin\varphi_\nu = \frac{\tan\alpha_1}{\tanh|a_1|}\sin\varphi_\nu\\
&\tan\nu\frac{\vartheta_\nu}{2} = \tan\frac{90^\circ-\alpha_\nu}{2}\tan\frac{90^\circ-\varphi_\nu}{2}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (171)$$

where

$$\left.\begin{aligned}
&\sin\alpha_\nu = \frac{\tan\alpha_1}{\tanh|a_1|\tan\bar{a}_\nu}\\
&\nu = \frac{\cos\bar{a}_\nu\sin\alpha_1}{\sinh|a_1|\tan\alpha_\nu}\\
&\bar{a}_\nu = 90^\circ - |\varkappa_\nu|\,|a_1|
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (172)$$

As the main curve, we define the curve obtained if $r_\nu - \bar{a}_\nu = r_\nu - (90^\circ - |\varkappa_\nu||a_1|)$ is chosen in the first or fourth quadrant, and as antipode curve, the curve obtained if $r_\nu - \bar{a}_\nu = r_\nu - (90^\circ - |\varkappa_\nu||a_1|)$ is chosen in the second or third quadrant. Then $\bar{a}_\nu$ becomes the mean radius vector of the main curve. If $\varphi_\nu = \varphi_1 = \varphi$, we obtain by eliminating φ from the first equations in (87) and (171)

$$\tan[r_1 - (90^\circ + i\,|a_1|)] = \tan(r_1 - \bar{a}_1) = \tan(r_\nu - \bar{a}_\nu)$$

As this corresponds to the angle $C_\nu = r_1 - r_\nu = \bar{a}_1 - \bar{a}_\nu$ between the centres of rotation, we see that $r_1 - \bar{a}_1$ and $r_\nu - \bar{a}_\nu$ must be chosen in the same quadrant, i.e. the curves must be either both main curves or both antipode curves. For the case where $\varphi_\nu = 180^\circ + \varphi_1$, corresponding to the angle between the centres of rotation $= \bar{a}_1 + \bar{a}_\nu - 180^\circ$, we find in a similar way that one of the curves must be a main curve, and the other an antipode curve. (Note that here we have $C_\nu = r_1 + r_\nu$, since the sign for r_ν has been changed in the derivation of eq. (105).)

If in eqs. (171) and (172) we put $\varkappa_\nu = 1$, i.e. $|\varkappa_\nu| = \frac{1}{i}$, we obtain the basic curve which is thus a twin curve. We always choose $a_\nu = |\varkappa_\nu||a_1| < 90°$, thus also $\bar{a}_\nu = 90° - |\varkappa_\nu||a_1| < < 90°$.

If the substitutions (170) are inserted in eqs. (110) and at the same time the bars in $\bar{\alpha}_\nu$ and $\bar{\alpha}_T$ are omitted, we again obtain the same equations, which thus are valid also here.

The equations (171) in parametric form are obtained by making the substitutions (86) and (170) in the corresponding equations (105) and (103) for rolling curves to the real spherical conic section. Thus, since eq. (109) is obtained by making the substitutions (106) in eq. (105), the equation in non-parametric form can be obtained by making the same substitutions, (86), (170) and (106), in eq. (109), i.e.

$$\left.\begin{array}{ll}
a_1 & \text{in the rolling curve to the real spherical conic section is replaced by } i\,|a_1| \\
e & \text{" " " " " " " " " " " " " " } i\,|e| \\
a_\nu & \text{" " " " " " " " " " " " " " } \bar{a}_\nu - 90° \\
\frac{\alpha_T}{2} & \text{" " " " " " " " " " " " " " } 90° - \frac{\alpha_T}{2} \\
\sin\alpha_1 & \text{" " " " " " " " " " " " " " } \frac{i}{\tan\alpha_1} \\
\cos\alpha_1 & \text{" " " " " " " " " " " " " " } \frac{1}{\sin\alpha_1} \\
\sin\varphi_1 & \text{" " " " " " " " " " " " " " } \frac{1}{\sin\varphi_1} \\
\cos\varphi_1 & \text{" " " " " " " " " " " " " " } \frac{i}{\tan\varphi_1} \\
\varkappa_\nu & \text{" " " " " " " " " " " " " " } i\,|\varkappa_\nu| \\
\sin\alpha_\nu & \text{" " " " " " " " " " " " " " } \frac{1}{\sin\alpha_\nu} \\
\cos\alpha_\nu & \text{" " " " " " " " " " " " " " } \frac{1}{i\tan\alpha_\nu} \\
\sin\varphi_\nu & \text{" " " " " " " " " " " " " " } \frac{1}{\sin\varphi_\nu} \\
\cos\varphi_\nu & \text{" " " " " " " " " " " " " " } \frac{i}{\tan\varphi_\nu}
\end{array}\right\} \quad (173)$$

This gives the equation in non-parametric form

$$\tan r_\nu = \frac{\sin\bar{a}_\nu \cos\bar{a}_\nu \cos^2\alpha_\nu}{\sin\alpha_\nu \left(\frac{1-\sin^2\bar{a}_\nu\cos^2\alpha_\nu}{\sin\alpha_\nu} - \cos\nu\vartheta_\nu\right)} \quad \ldots\ldots\ldots\ldots \quad (174)$$

The equation may also be obtained in the same form as eqs. (112) by inserting the substitutions (173) in eqs. (112)

$$
\left.\begin{aligned}
&\tan r_\nu = \frac{k_1}{k_2 - \cos \nu \vartheta_\nu} \quad (k_1 \text{ and } k_2 \text{ are positive})\\
&\tan 2\,\bar{a}_\nu = \frac{2\,k_1\,k_2}{k_2^2 - k_1^2 - 1}\\
&\sin \alpha_T = \frac{2\,k_2}{k_2^2 + k_1^2 + 1}\\
&\sin \alpha_\nu = \tan \frac{\alpha_T}{2}\\
&\sin \alpha_1 = \tan \alpha_\nu \sqrt{\frac{\sin^2 \bar{a}_\nu \cos^2 \alpha_\nu - \nu^2}{1 - \sin^2 \bar{a}_\nu \cos^2 \alpha_\nu}} \text{ or } \cos \alpha_1 = \frac{1}{\cos \alpha_\nu} \sqrt{\frac{\cos^2 \bar{a}_\nu + \nu^2 \tan^2 \alpha_\nu}{\cos^2 \bar{a}_\nu + \tan^2 \alpha_\nu}}\\
&\sinh |a_1| = \frac{\cos \bar{a}_\nu \sin \alpha_1}{\nu \tan \alpha_\nu} \text{ or } \cosh |a_1| = \frac{\sin \bar{a}_\nu \cos \alpha_1 \cos \alpha_\nu}{\nu}\\
&\left(\frac{\tanh |a_1|}{\tan \alpha_1} = \frac{1}{\tan \bar{a}_\nu \sin \alpha_\nu} = \tan |e|\,|a_1|\right)
\end{aligned}\right\} \quad (175)
$$

This system of equations is used if we have an equation given in the same form as the first equation in the system. By using the following equations, it is then possible to determine α_ν, $\bar{a}_\nu = 90° - |\varkappa_\nu||a_1|$, α_1 and $|a_1|$, after which the equation may be expressed in the forms (171) or (174).

We always choose α_T, α_ν, α_1, a_1 and $\bar{a}_\nu$ (consequently also $|\varkappa_\nu||a_1|$) in the first quadrant.

The condition for obtaining real solutions from eqs. (175) is that

$$\nu \leq \sin \bar{a}_\nu \cos \alpha_\nu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (176)$$

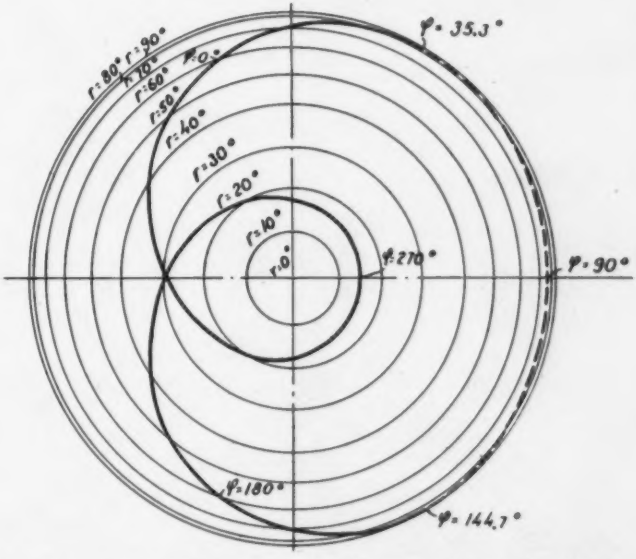


Fig. 9. Rolling curve (main curve) to the imaginary spherical hyperbola having $\bar{a}_\nu = 90° - |\varkappa_\nu||a_1| = 60°$, $\tanh |a_1| = \frac{1}{\sqrt{3}}$, $\alpha_1 = 30°$ and $\nu = \frac{1}{2}$.

As the first four equations in (112) and (175) are the same, with the exception that $a_\nu = |\varkappa_\nu||a_1|$ in (112) is replaced by $\bar{a}_\nu$ in (175), it can be understood from the conditions (113) and (176) that, if the system of equations (112) does not give real solutions, then instead (175) will give real solutions. A curve having the same equation as the first equation in (112) or (175) is accordingly a rolling curve with a real ν-value, either to the real spherical conic section or to the imaginary spherical hyperbola. The limiting case where $\nu = \sin \bar{a}_\nu \cos \alpha_\nu$, may also be considered as a rolling curve to the real spherical conic section and in the following it is also treated as such a rolling curve.

Fig. 9 shows the main curve of a rolling curve to the imaginary spherical hyperbola having $\tanh |a_1| = \frac{1}{\sqrt{3}}$, $\alpha_1 = 30°$, $\nu = \frac{1}{2}$ and $|\varkappa_\nu||a_1| = 30°$, i.e. $\bar{a}_\nu = 60°$.

2) *Rolling curves with imaginary periodic number*

If in eqs. (172) and (175) $\tan \bar{a}_\nu < \dfrac{\tan \alpha_1}{\tanh |a_1|} = \tan |e||a_1|$, i.e. $|\varkappa_\nu| > |e|$, we then have $\sin \alpha_\nu > 1$ and $\cos \alpha_\nu$ and ν imaginary. We can make here the substitutions (173) in the equations (115), (116) and (117) of the rolling curve with an imaginary ν-value to the real spherical conic section and thus obtain the equation of the rolling curve in parametric form

$$\left.\begin{aligned} \tan (r_\nu - \bar{a}_\nu) &= \frac{\tan \bar{a}_\nu}{\sin \alpha_\nu} \sin \varphi_\nu = \frac{\tan \alpha_1}{\tanh |a_1|} \sin \varphi_\nu \\ \tanh |\nu| \frac{\vartheta_\nu}{2} &= \tan \frac{90° \mp \alpha_\nu}{2} \tan \frac{90° \mp \varphi_\nu}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (177)$$

where

$$\left.\begin{aligned} \sin \alpha_\nu &= \frac{\tan \bar{a}_\nu \tanh |a_1|}{\tan \alpha_1} \\ |\nu| &= \frac{\cos \bar{a}_\nu \sin \alpha_1 \cos \alpha_\nu}{\sinh |a_1|} \\ \bar{a}_\nu &= 90° - |\varkappa_\nu| |a_1| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (178)$$

and the equation in non-parametric form

$$\tan r_\nu = - \frac{\sin \bar{a}_\nu \cos \bar{a}_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu \left(\dfrac{1 - \cos^2 \bar{a}_\nu \cos^2 \alpha_\nu}{\sin \alpha_\nu} \mp \cosh |\nu| \vartheta_\nu \right)} \quad \ldots\ldots\ldots \quad (179)$$

From the second equation in (177) it can be seen that the upper signs give a real solution for ϑ_ν only if

$$\tan \frac{90° - \alpha_\nu}{2} \left| \tan \frac{90° - \varphi_\nu}{2} \right| \leq 1$$

or

$$\tan \frac{90° - \alpha_\nu}{2} \leq \left| \tan \frac{90° + \varphi_\nu}{2} \right|$$

or

$$- \alpha_\nu \leq \varphi_\nu \leq \alpha_\nu + 180°$$

In a similar way, it can be shown that the lower signs give a real solution only if

$$-\alpha_\nu \geq \varphi_\nu \geq 180^\circ + \alpha_\nu$$

The two cases correspond to two different branches. For the main curves both branches approach the origin spirally. The upper signs correspond to the larger branch. For the main curve r_ν will then always become positive, which is seen from eqs. (177), if the limiting values for φ_ν are inserted. The lower signs correspond to the smaller branch in which case r_ν always becomes negative.

The equation of the curve with an imaginary ν-value can be obtained in the same form as eqs. (175) by inserting the substitutions (173) in the corresponding equations (118) for the rolling curve with an imaginary ν-value to the real spherical conic section. We obtain

$$\left.\begin{aligned}
&\tan r_\nu = \frac{k_1}{k_2 - \cosh |\nu| \vartheta_\nu} = \frac{\mp |k_1|}{\pm |k_2| - \cosh |\nu| \vartheta_\nu} \\
&\tan 2\,\bar{a}_\nu = -\frac{2\,k_1\,k_2}{k_1^2 - k_2^2 + 1} = \frac{2\,|k_1|\,|k_2|}{k_1^2 - k_2^2 + 1} \\
&\sin \alpha_T = \frac{2\,|k_2|}{k_2^2 + k_1^2 + 1} \\
&\sin \alpha_\nu = \tan \frac{\alpha_T}{2} \\
&\sin \alpha_1 = \frac{1}{\cos \alpha_\nu} \sqrt{\frac{\sin^2 \bar{a}_\nu / \tan^2 \alpha_\nu - |\nu|^2}{\sin^2 \bar{a}_\nu / \tan^2 \alpha_\nu + 1}} \text{ or } \cos \alpha_1 = \tan \alpha_\nu \sqrt{\frac{|\nu|^2 + \cos^2 \bar{a}_\nu \cos^2 \alpha_\nu}{1 - \cos^2 \bar{a}_\nu \cos^2 \alpha_\nu}} \\
&\sinh |a_1| = \frac{\cos \bar{a}_\nu \sin \alpha_1 \cos \alpha_\nu}{|\nu|} \text{ or } \cosh |a_1| = \frac{\sin \bar{a}_\nu \cos \alpha_1}{|\nu| \tan \alpha_\nu} \\
&\left(\frac{\tanh |a_1|}{\tan \alpha_1} = \frac{\sin \alpha_\nu}{\tan \bar{a}_\nu} = \tan |e|\,|a_1|\right)
\end{aligned}\right\} \quad (180)$$

By comparing this with eq. (179), we see that we must have either k_1 negative and k_2 positive, corresponding to the larger branch and the upper signs in the first equation or k_1 positive and k_2 negative, corresponding to the smaller branch and the lower signs in the first equation.

We always choose α_T, α_ν, α_1 and $|\varkappa_\nu||a_1|$; thus also $\bar{a}_\nu = 90^\circ - |\varkappa_\nu||a_1|$ in the first quadrant. In the same way as for the system of equations (118), it can be shown that the condition for obtaining real solutions is that

$$|\nu| \leq \frac{\sin \bar{a}_\nu}{\tan \alpha_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (181)$$

As the four first equations in (118) and (180) are the same except that $a_\nu = \varkappa_\nu a_1$ in eqs. (118) is replaced by $\bar{a}_\nu = 90^\circ - |\varkappa_\nu||a_1|$ in eqs. (180), it can be understood from the conditions (119) and (181) that, if the system of equations (118) does not give real solutions, then instead (180) gives real solutions. A curve having an equation of the same type as the first equation in (118) or (180) is, therefore, always a rolling curve with an imaginary ν-value either to the real spherical conic section or to the imaginary spherical hyperbola.

3) *Special cases of rolling curves to the imaginary spherical hyperbola*

α) *The crown-wheel curve (the spherical cos-curve having* $\nu < 1$)

From eqs. (171) it can be seen that the geodetic mean radius of the rolling curve $= 90° - |\varkappa_\nu||a_1| = \bar{a}_\nu$. The equation of the crown-wheel curve can therefore be obtained if we put $\bar{a}_\nu = 90°$, i.e. $|\varkappa_\nu||a_1| = 0$. From eqs. (172) we then obtain

$$\left.\begin{aligned} &\sin \alpha_\nu = 0;\ \text{thus } \alpha_\nu = 0 \\ &\nu = \frac{\cos \bar{a}_\nu \sin \alpha_1}{\sinh |a_1| \tan \alpha_\nu} = \frac{\sin \bar{a}_\nu \cos \alpha_1 \cos \alpha_\nu}{\cosh |a_1|} = \frac{\cos \alpha_1}{\cosh |a_1|} \end{aligned}\right\}$$

and the systems of equations (171) and (172) take the form

$$\left.\begin{aligned} &\tan (r_\nu - 90°) = \frac{\tan \alpha_1}{\tanh |a_1|} \sin \varphi_\nu \\ &\nu \vartheta_\nu = 90° - \varphi_\nu \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (182)$$

where

$$\left.\begin{aligned} &\alpha_\nu = 0 \\ &\nu = \frac{\cos \alpha_1}{\cosh |a_1|} \\ &\bar{a}_\nu = 90° - |\varkappa_\nu|\,|a_1| = 90° \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (183)$$

which are the equations of the crown-wheel curve in parametric form. The equation in non-parametric form is obtained if φ_ν is eliminated from eqs. (182)

$$\tan r_\nu = -\frac{\tanh |a_1|}{\tan \alpha_1 \cos \nu \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (184)$$

If we project radially the curve on to a cylinder tangential to the great circle $r = 90°$, and if we place the x-axis on the great circle, and the y-axis along a generatrix of the cylinder, then, because $y = \tan (90° - r_\nu)$, the equation of the radially projected curve becomes

$$y \frac{\tanh |a_1|}{\tan \alpha_1} = -\cos \nu \vartheta_\nu = -\cos \left(\frac{\cos \alpha_1}{\cosh |a_1|} x \right)$$

For this reason the curve is designated the spherical cos-curve.

The crown-wheel curve now treated is included in the system of equations (175) if $k_2 \to +0$. From the third equation in (175) we then obtain $\sin \alpha_T \to +0$, i.e. $\alpha_T = 0$, which inserted in the fourth equation gives $\alpha_\nu = 0$. From the second equation we further obtain $\tan 2\,\bar{a}_\nu \to -0$, thus

$$\bar{a}_\nu \to 90°$$

The fifth equation in (175) gives an indeterminate value of α_1. The limiting value of α_1 can, however, be obtained if the fifth equation is replaced by the following expression obtained from eq. (184)

$$k_1 = \frac{\tanh |a_1|}{\tan \alpha_1}$$

Eqs. (175) now take the form

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\cos \nu\vartheta_\nu} \\ &\alpha_\nu = 0 \\ &\bar{a}_\nu = 90^\circ \\ &\tanh |a_1| = k_1 \tan \alpha_1 \\ &\cosh |a_1| = \frac{\cos \alpha_1}{\nu} \end{aligned}\right\}$$

Eliminating $|a_1|$ from the last two equations gives

$$\cos \alpha_1 = \sqrt{\frac{\nu^2 + k_1^2}{1 + k_1^2}}$$

The above system of equations can now be written in the form

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{-\cos \nu\vartheta_\nu} \quad (k_1 \text{ is positive}) \\ &\bar{a}_\nu = 90^\circ \\ &\alpha_\nu = 0 \\ &\cos \alpha_1 = \sqrt{\frac{\nu^2 + k_1^2}{1 + k_1^2}} \\ &\cosh |a_1| = \frac{\cos \alpha_1}{\nu} = \frac{1}{\nu}\sqrt{\frac{\nu^2 + k_1^2}{1 + k_1^2}} \\ &\left(\frac{\tanh |a_1|}{\tan \alpha_1} = \tan |e||a_1| = k_1\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (185)$$

The above system of equations is used for determining $|a_1|$ and α_1 if the equation of the curve is given in the same form as the first equation in the system. The equation of the curve can then be written down in the same form as eqs. (182) or (184).

The condition for obtaining real solutions from eqs. (185) is that

$$\nu^2 + k_1^2 \leq 1 + k_1^2 \text{ and } \nu^2 + k_1^2 \geq \nu^2 (1 + k_1^2)$$

or

$$\nu = \frac{\cos \alpha_1}{\cosh |a_1|} \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (186)$$

Eqs. (182) and (184) thus only comprise the spherical cos-curves for which $\nu \leq 1$. The remaining spherical cos-curves having $\nu \geq 1$ are rolling curves to the real spherical conic section, as may be seen from eqs. (123) and (124).

The limiting case where $\nu = 1$ is also a limiting case to the crown-wheel curve to the real spherical conic section and in the following it is treated as such a curve according to eqs. (120), (122) and (123).

β) The rolling curve with $|\varkappa_\nu||a_1| = 90^\circ$ *(the spherical cosh-curve having* $|\nu| < |\tan r_{\nu\max}|$*)*

If in eqs. (178) we put $|\varkappa_\nu||a_1| = 90^\circ$, i.e. $\bar{a}_\nu = 0$, we obtain

$$\alpha_\nu = 0$$

The systems of equations (177) and (178) then take the form

$$\left.\begin{aligned} \tan r_\nu &= \frac{\tan \alpha_1}{\tanh |a_1|} \sin \varphi_\nu \\ \tanh |\nu| \frac{\vartheta_\nu}{2} &= \tan \frac{90^\circ \mp \varphi_\nu}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (187)$$

where

$$\left.\begin{aligned} \alpha_\nu &= 0 \\ |\nu| &= \frac{\sin \alpha_1}{\sinh |a_1|} \\ \bar{a}_\nu &= 90^\circ - |\varkappa_\nu|\,|a_1| = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (188)$$

which is the equation of the curve in parametric form.

The equation in non-parametric form is obtained if the above values of $\alpha_\nu = 0$ and $|\nu|$ are inserted in eq. (179)

$$\tan r_\nu = \pm \frac{\tan \alpha_1}{\tanh |a_1| \cosh |\nu| \vartheta_\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (189)$$

If we project radially the curve on to a cylinder tangential to the great circle $r = 90^\circ$, and if we place the x-axis on the great circle, and the y-axis along a generatrix of the cylinder, then, because $y = \tan (90^\circ - r_\nu)$, the equation of the radially projected curve becomes

$$y \frac{\tan \alpha_1}{\tanh |a_1|} = \pm \cosh |\nu| \vartheta_\nu = \pm \cosh \left(\frac{\sin \alpha_1}{\sinh |a_1|} x \right)$$

For this reason the curve is designated the spherical cosh-curve.

The curve treated now is included in the system of equations (180), in which case $|k_2| \to 0$. From the third equation in (180) we then obtain $\alpha_T \to 0$, which inserted in the fourth equation gives $\alpha_\nu = 0$. From the second equation in (180) we further obtain $2\,\bar{a}_\nu \to +0$. The fifth equation gives an indeterminate value of α_1. The limiting value of α_1 can, however, according to eq. (189), be obtained from the expression

$$|k_1| = \frac{\tan \alpha_1}{\tanh |a_1|}$$

By using the sixth equation in (180) which here reduces to

$$\sinh |a_1| = \frac{\sin \alpha_1}{|\nu|}$$

and eliminating $|a_1|$, we obtain

$$\cos \alpha_1 = \sqrt{\frac{1 + |\nu|^2}{1 + k_1^2}}$$

If this expression and the values of $|k_2|$, α_ν and $\bar{a}_\nu$ obtained above are now inserted in eqs. (180), we obtain

$$\left.\begin{aligned}
&\tan r_\nu = \frac{k_1}{-\cosh|\nu|\vartheta_\nu} = \frac{\mp|k_1|}{-\cosh|\nu|\vartheta_\nu}\\
&\bar{a}_\nu = 0\\
&\alpha_\nu = 0\\
&\cos\alpha_1 = \sqrt{\frac{1+|\nu|^2}{1+k_1^2}} \text{ or } \sin\alpha_1 = \sqrt{\frac{k_1^2-|\nu|^2}{1+k_1^2}}\\
&\sinh|a_1| = \frac{\sin\alpha_1}{|\nu|}\\
&\left(\frac{\tanh|a_1|}{\tan\alpha_1} = \frac{1}{|k_1|} = \tan|e||a_1|\right)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (190)$$

where the + sign applies if k_1 is positive and the — sign if k_1 is negative. The above system of equations is used for determining $|a_1|$ and α_1 if the curve is given in the same form as the first equation in the system. The equation of the curve can now be written in the same form as eqs. (187) and (189). In the case where $|\nu| = k_1$, we obtain $\alpha_1 = 0$ and $|a_1| = 0$. By using eq. (189) we observe then that $\dfrac{\tan\alpha_1}{\tanh|a_1|} = |k_1|$. This limiting case is, however, also a limiting case of the rolling curves with an imaginary ν-value having $\varkappa_\nu a_1 = 0$ to the real spherical conic section and will be treated in the following as such a curve according to eqs. (128).

The condition for obtaining real solutions from eqs. (190) is that

$$1 + |\nu|^2 < 1 + k_1^2$$

or

$$|\nu| \leq \frac{\tan\alpha_1}{\tanh|a_1|} = \tan|r_{\nu\max}| = |k_1| \quad \ldots\ldots\ldots\ldots\ldots \quad (191)$$

The remaining cosh-curves are rolling curves to the real spherical conic section, as can be seen from eqs. (128) and (129).

If

$$|\nu| = \frac{\tan\alpha_1}{\tanh|a_1|}$$

we have a limiting case between the two types of curve, which in the following is treated as a rolling curve to the real spherical conic section according to eqs. (128).

γ) *The rolling curve with periodic number* $\nu = 0$

If in eqs. (171) and (172) we let $\nu \to 0$, then, from the second equation in (172), we find that $\tan\alpha_\nu = \tan\alpha_0 \to \infty$; thus $\alpha_0 = 90°$.

The first equation in (172) now gives

$$\tan\bar{a}_0 = \frac{\tan\alpha_1}{\tanh|a_1|} = \frac{1}{\tan|e||a_1|}, \text{ i.e. } |\varkappa_0| = e$$

and the second equation in (171) takes the form

$$\tan \nu \frac{\vartheta_0}{2} = \frac{1 - \sin \alpha_0}{\cos \alpha_0} \tan \frac{90^\circ - \varphi_0}{2}$$

or, as $\nu \vartheta_0 \to 0$ if $\nu = 0$,

$$\vartheta_0 = \frac{2(1 - \sin \alpha_0)}{\nu \cos \alpha_0} \tan \frac{90^\circ - \varphi_0}{2}$$

Substituting ν from the second equation in (172) gives

$$\vartheta_0 = \frac{2(1 - \sin \alpha_0) \sinh |a_1| \tan \alpha_0}{\cos \bar{a}_0 \sin \alpha_1 \cos \alpha_0} \tan \frac{90^\circ - \varphi_0}{2} =$$

$$= \frac{2}{1 + \sin \alpha_0} \cdot \frac{\sinh |a_1| \sin \alpha_0}{\cos \bar{a}_0 \sin \alpha_1} \tan \frac{90^\circ - \varphi_0}{2}$$

which, as $\sin \alpha_0 \to 1$, and using the first equation in (172), gives

$$\vartheta_0 = \frac{\cosh |a_1|}{\sin \bar{a}_0 \cos \alpha_1} \tan \frac{90^\circ - \varphi_0}{2}$$

The equation of the rolling curve in parametric form thus becomes

$$\left.\begin{aligned} \tan (r_0 - \bar{a}_0) &= \tan \bar{a}_0 \sin \varphi_0 = \frac{\tan \alpha_1}{\tanh |a_1|} \sin \varphi_0 \\ \vartheta_0 &= \frac{\cosh |a_1|}{\sin \bar{a}_0 \cos \alpha_1} \tan \frac{90^\circ - \varphi_0}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (192)$$

where

$$\left.\begin{aligned} \alpha_0 &= 90^\circ \\ \nu &= 0 \\ \bar{a}_0 &= 90^\circ - |\varkappa_0|\, |a_1| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (193)$$

The equation of the rolling curve in non-parametric form can be obtained by eliminating φ_0 from (192) in the following way.

From the first equation we solve with respect to $\sin \varphi_0$

$$\sin \varphi_0 = \frac{\cot \bar{a}_0 \tan r_0 - 1}{\tan \bar{a}_0 \tan r_0 + 1}$$

and obtain

$$\tan^2 \frac{90^\circ - \varphi_0}{2} = \frac{1 - \sin \varphi_0}{1 + \sin \varphi_0} = \sin^2 \bar{a}_0 \frac{2 \cot \bar{a}_0 + \tan r_0 (1 - \cot^2 \bar{a}_0)}{\tan r_0}$$

which, substituted in the second equation, gives

$$\vartheta_0^2 = \frac{\cosh^2 |a_1| \, [2 \cot \bar{a}_0 + \tan r_0 (1 - \cot^2 \bar{a}_0)]}{\cos^2 \alpha_1 \tan r_0}$$

Solving this with respect to $\tan r_0$ and using the first equations in (172) and (193), we obtain the equation of the rolling curve in non-parametric form

$$\tan r_0 = \frac{2 \tan \bar{a}_0}{1 - \tan^2 \bar{a}_0 + \cos^2 \alpha_1 (\tan^2 \bar{a}_0 - \tan^2 \alpha_1) \vartheta_0^2} \quad \ldots\ldots\ldots \quad (194)$$

where ϑ_0 is measured in radians.

The equation in non-parametric form may also be written as

where

$$\left.\begin{aligned} \tan r_0 &= \frac{k_1}{k_2 + \vartheta_0^2} \\ k_1 &= \frac{2 \tan \bar{a}_0}{\cos^2 \alpha_1 (\tan^2 \bar{a}_0 - \tan^2 \alpha_1)} \\ k_2 &= \frac{1 - \tan^2 \bar{a}_0}{\cos^2 \alpha_1 (\tan^2 \bar{a}_0 - \tan^2 \alpha_1)} \\ \tan \bar{a}_0 &= \frac{1}{\tan |\varkappa_0||a_1|} = \frac{\tan \alpha_1}{\tanh |a_1|} \end{aligned}\right\}$$

By dividing the second of these equations by the third, we obtain

$$\frac{k_1}{k_2} = \frac{2 \tan \bar{a}_0}{1 - \tan^2 \bar{a}_0} = \tan 2 \bar{a}_0$$

From the second equation in the above system of equations we can solve with respect to cos α_1, obtaining

$$\cos \alpha_1 = \cos \bar{a}_0 \sqrt{\frac{k_1 + 2 \tan \bar{a}_0}{k_1}} \text{ or } \tan \alpha_1 = \tan \bar{a}_0 \sqrt{\frac{k_1 - 2/\tan \bar{a}_0}{k_1 + 2 \tan \bar{a}_0}}$$

The above system of equations can now be rewritten as

$$\left.\begin{aligned} &\tan r_0 = \frac{k_1}{k_2 + \vartheta_0^2} \text{ (k_1 is positive)} \\ &\tan 2 \bar{a}_0 = \frac{k_1}{k_2} \\ &\varkappa_0 = 90^\circ \\ &\tan \alpha_1 = \tan \bar{a}_0 \sqrt{\frac{k_1 - 2/\tan \bar{a}_0}{2 \tan \bar{a}_0 + k_1}} \\ &\tanh |a_1| = \frac{\tan \alpha_1}{\tan \bar{a}_0} \\ &\left(\frac{\tanh |a_1|}{\tan \alpha_1} = \frac{1}{\tan \bar{a}_0} = \tan |\epsilon||a_1|\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (195)$$

The condition for obtaining real solutions from eqs. (195) is that tan α_1 becomes real and tanh $|a_1| \leq 1$. As k_1 is positive, the first condition implies that

$$k_1 \geq 2/\tan \bar{a}_0$$

From the fourth equation it can be seen that we always have tan $\alpha_1 \leq \tan \bar{a}_0$. As α_1 and $\bar{a}_0$ both lie in the first quadrant, we thus always have tanh $|a_1| \leq 1$.

If we form $\tan 2\,\bar{a}_0$ from the above condition, we obtain by using the second equation in (195)

$$\tan 2\,\bar{a}_0 = \frac{k_1}{k_2} \geq \frac{4/k_1}{1 - 4/k_1^2}, \text{ if } k_2 \text{ and } (1 - 4/k_1^2) \text{ have the same signs}$$

or

$$\frac{k_1}{k_2} \leq \frac{4/k_1}{1 - 4/k_1^2}, \text{ if } k_2 \text{ and } (1 - 4/k_1^2) \text{ have different signs}$$

Thus, in both cases

$$\frac{k_1^2}{4} - k_2 \geq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (196)$$

By comparing the systems of equations (133) and (195), we find that the first equation is the same in the two systems. Further, if we compare the conditions (134) and (196), it can be seen that a rolling curve of the type given by the first equation in (133) and (195) is a rolling curve either to the real spherical conic section or to the imaginary spherical hyperbola. The limiting case between these two types of rolling curve, where $k_1^2/4 - k_2 = 1$ is treated in the following as a rolling curve to the real spherical conic section.

b) The combination of rolling curves to the imaginary spherical hyperbola to form rolling curve pairs and the angles between the centres of rotation involved

From the derivations of the rolling curves it can be seen that all rolling curves to the imaginary spherical hyperbola with the same $|a_1|$- and α_1-values as the basic hyperbola can form rolling curve pairs with the basic hyperbola with two different angles between the centres of rotation; (a) with one angle $C_\nu = \bar{a}_1 - \bar{a}_\nu = |a_1|\,(i + |\varkappa_\nu|)$, in which case both curves are main curves and $\varphi_\nu = \varphi_1$, (b) with another angle between the centres of rotation $C_\nu = \bar{a}_1 + \bar{a}_\nu - 180° = |a_1|\,(i - |\varkappa_\nu|)$, in which case $\varphi_\nu = 180° + \varphi_1$ and the basic hyperbola is a main curve, and the rolling curve an antipode curve. Applying the same reasoning as was used for elliptical rolling curves, i.e., if we imagine two hyperbolic rolling curves rolling simultaneously on the same main basic hyperbola in such a manner that the point of contact is common for all three curves, we easily find that two hyperbolic rolling curves with the same $|a_1|$- and α_1-values can also form rolling curve pairs with each other with two different angles between the centres of rotation; namely

$$\left.\begin{aligned} C_{L-R} &= \bar{a}_{\nu L} - \bar{a}_{\nu R} = |a_1|\,(|\varkappa_R| - |\varkappa_L|), \text{ in which case } \varphi_R = \varphi_L \text{ and both curves} \\ &\qquad \text{are main curves or antipode curves} \\ C_{L-R} &= \bar{a}_{\nu L} + \bar{a}_{\nu R} - 180° = -|a_1|\,(|\varkappa_R| + |\varkappa_L|), \text{ in which case } \varphi_R = 180° + \varphi_L \\ &\qquad \text{and one curve is a main curve and the other an antipode curve} \end{aligned}\right\} \quad (197)$$

For spherical rolling curves, however, it is possible to replace the assumed centre of rotation for any of the rolling curves by its antipode. For rolling curves to the imaginary spherical hyperbola which are symmetrical with respect to the polar axis, this, however, can be done by replacing a main curve by its antipode curve or vice versa. At the same time, however, the φ_ν-value valid for the original curve must be replaced by the value $180° - \varphi_\nu$. This is obvious from the fact that the point of contact moves in the opposite direction on the antipode curve,

and that for a point of contact corresponding to $\varphi_\nu = 90°$ for the main curve we must have the same value of φ_ν for the antipode curve. Further, because eqs. (137) and (138) apply also to rolling curves to the imaginary spherical hyperbola, we can easily find the following possible different combinations of rolling curves to the imaginary spherical hyperbola, where the absolute value of the angle between the centres of rotation $|C_{L-R}|$ is always taken $< 180°$.

Rolling case 1

where both curves are main curves or antipode curves

$$\left.\begin{aligned} &\varphi_R = \varphi_L \\ &C_{L-R} = \bar{a}_{\nu L} - \bar{a}_{\nu R} = |a_1|\,(|\varkappa_R| - |\varkappa_L|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (198)$$

Rolling case 2

where both curves are main curves or antipode curves

$$\left.\begin{aligned} &\varphi_R = -\varphi_L \\ &C_{L-R} = \bar{a}_{\nu L} + \bar{a}_{\nu R} = 180° - |a_1|\,(|\varkappa_R| + |\varkappa_L|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (199)$$

Rolling case 3

where one curve is a main curve and the other an antipode curve

$$\left.\begin{aligned} &\varphi_R = 180° - \varphi_L \\ &C_{L-R} = \bar{a}_{\nu L} - \bar{a}_{\nu R} \mp 180° = |a_1|\,(|\varkappa_R| - |\varkappa_L|) \mp 180° \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (200)$$

where the upper sign is valid if $\bar{a}_{\nu L} > \bar{a}_{\nu R}$ and vice versa.

Rolling case 4

where one of the curves is a main curve and the other an antipode curve

$$\left.\begin{aligned} &\varphi_R = 180° + \varphi_L \\ &C_{L-R} = \bar{a}_{\nu L} + \bar{a}_{\nu R} - 180° = -|a_1|\,(|\varkappa_R| + |\varkappa_L|) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (201)$$

In the same way as for rolling curves to the real conic section, we can find that for a given value of C_{L-R} the last four systems of equations give one, but only one, solution for $0 < \bar{a}_\nu < 90°$. Therefore, for a given value of C_{L-R} we have one, but only one, mating curve to a given rolling curve.

To compile the combinations of rolling curves which give external meshing with an absolute value of the angle between the centres of rotation $< 180°$, we can apply the same reasoning as was used for rolling curves to the real spherical conic section, i.e., we can investigate the nature of meshing if a main rolling curve rolls on another main rolling curve from $\varphi = 0$ to $\varphi = 360°$. The nature of meshing can only be altered, and is always altered, when the radius vector of one of the rolling curves changes its sign, i.e., when a larger branch with an imaginary ν-value changes to a smaller branch or vice versa. It is, therefore, easy to determine the nature of meshing during the rolling, considering that the curve which either has a real value of ν or is a larger branch with an imaginary ν-value, gives

internal meshing for the distance $\bar{a}_{\nu L} - \bar{a}_{\nu R}$ between the centres of rotation and external meshing if this distance is $\bar{a}_{\nu L} + \bar{a}_{\nu R}$. In the cases where during the rolling we obtain internal meshing, the internally meshed curve is replaced by its antipode curve, in which case also the relation between φ_R and φ_L and the distance C_{L-R} between the centres of rotation are altered according to the equations and laws given above. We then obtain external meshing. In this way, we can obtain all the possible combinations giving external meshing, and having $|C_{L-R}| < 180°$, which are tabulated below.

The one rolling curve	The other rolling curve	Rolling case	At point of contact $\varphi_R =$
ν real, main curve ν imaginary, larger branch of main curve	ν imaginary, smaller branch of main curve	1	$+\varphi_L$
	ν real, main curve ν imaginary, larger branch of main curve	2	$-\varphi_L$
ν imaginary, smaller branch of main curve	ν imaginary, smaller branch of antipode curve		
ν real, antipode curve ν imaginary, larger branch of antipode curve	ν real, main curve ν imaginary, larger branch of main curve	3	$180° - \varphi_L$
	ν imaginary, smaller branch of main curve	4	$180° + \varphi_L$

D. Rolling curves to the limiting curve for the imaginary spherical conic sections when the eccentricity $e \to 0$ and $\vartheta_\nu \to \pm\infty$

a) Deduction of the equations of the rolling curves

1) *Normal rolling curves*

The equation of the basic curve in parametric form is given by eqs. (97).

Introducing the substitutions (compare eqs. (101))

$$\left.\begin{aligned} a_\nu &= |\varkappa_\nu|\,|a_1| \\ C_\nu &= |a_1|\,(i \mp |\varkappa_\nu|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (202)$$

into eqs. (7) which are generally valid, and considering that the angle of displacement γ_1 equals the polar angle ϑ_1, we obtain

$$\frac{d\gamma_\nu}{d\vartheta_1} = \frac{\sin r_1}{\sin\,[r_1 - |a_1|\,(i \mp |\varkappa_\nu|)]} =$$

$$= \frac{\sin r_1}{\sin\,(r_1 - i\,|a_1|)\cos\,(|\varkappa_\nu|\,|a_1|) \pm \cos\,(r_1 - i\,|a_1|)\sin\,(|\varkappa_\nu|\,|a_1|)}$$

If values belonging to the basic curve are assigned the suffix 1, the first equation in (97) gives

$$\sin\,(r_1 - i\,|a_1|) = \frac{\sinh\,|a_1|\,\sin\varphi_1}{\pm\sqrt{\sinh^2|a_1| + \cos^2\varphi_1}}$$

and

$$\cos(r_1 - i|a_1|) = \frac{\cosh|a_1|\cos\varphi_1}{\pm\sqrt{\sinh^2|a_1| + \cos^2\varphi_1}}$$

The positive root is valid for the main curve because $(r - i|a_1|)$ and φ_1 lie in the same quadrant.

Further, from the second equation in (97) we obtain

$$\frac{d\vartheta_1}{d\varphi_1} = \mp\frac{1}{\sin\varphi_1\cos\varphi_1(i + \tan\varphi_1)}$$

Expanding $\tan(r - i|a_1|)$ in the first equation of (97) and solving with respect to $\tan r_1$ give

$$\tan r_1 = \frac{\tanh|a_1|(i + \tan\varphi_1)}{1 - i\tanh^2|a_1|\tan\varphi_1}$$

and

$$\sin r_1 = \frac{(i + \tan\varphi_1)\sinh|a_1|\cosh|a_1|\cos\varphi_1}{\pm\sqrt{\sinh^2|a_1| + \cos^2\varphi_1}}$$

If we now substitute these expressions for $\frac{d\vartheta_1}{d\varphi_1}$, $\sin(r_1 - i|a_1|)$, $\cos(r_1 - i|a_1|)$ and $\sin r_1$ in the above expression for $\frac{d\gamma_r}{d\vartheta_1}$, we obtain for rolling on the main curve of the basic curve

$$\mp d\gamma_r = \frac{\sinh|a_1|}{\pm[\cos(|\varkappa_r||a_1|)\tanh|a_1|\tan\varphi_1 \pm \sin(|\varkappa_r||a_1|)]\sin\varphi_1\cos\varphi_1}\,d\varphi_1$$

where the $\mp$ sign before $d\gamma_r$ originates from the expression for $\frac{d\vartheta_1}{d\varphi_1}$, i.e., the $-$ sign corresponds to rolling on the A-branch of the basic curve and the $+$ sign corresponds to rolling on the B-branch. The $\pm$ sign before the denominator originates from the expression for $\sin r_1$ and finally the $\pm$ sign before $\sin(|\varkappa_r||a_1|)$ refers to the distance between the centres of rotation. The upper sign corresponds to the upper sign in eqs. (202) and vice versa.

For rolling curves to the real spherical conic section, of which the rolling curves here in question are a special case, the relations (102) are valid. Here, however, we have $e = 0$ and thus $\sin\alpha_1 = \frac{\tan e\,a_1}{\tan a_1} = 0$. Instead of (102) we introduce therefore a corresponding substitution

$$\left.\begin{aligned}\tan\alpha_r &= \frac{\tan(|\varkappa_r||a_1|)}{\tanh|a_1|}\\ |\nu| &= \frac{\sin(|\varkappa_r||a_1|)}{\sinh|a_1|}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (203)$$

The above expression for $d\gamma_r$ can then be rewritten as

$$\mp|\nu|\,d\gamma_r = \frac{\tan\alpha_r}{(\tan\varphi_1 \pm \tan\alpha_r)\sin\varphi_1\cos\varphi_1}\,d\varphi_1$$

or

$$\mp|\nu|\,d\gamma_r = \frac{d\varphi_1}{\sin\varphi_1\cos\varphi} - \frac{1/\cos^2\varphi_1}{\tan\varphi_1 \pm \tan\alpha_r}\,d\varphi_1$$

or, if γ_ν is measured in radians, after integration

$$\pm e^{\mp |\nu| \gamma_\nu} = C_1 \frac{\tan \varphi_1}{\tan \alpha_\nu \pm \tan \varphi_1}$$

Since the polar angle $\vartheta_\nu = \gamma_\nu +$ a constant, and if we let

$$C_1 = \frac{1}{\cos^2 (|\varkappa_\nu| |a_1|)}$$

the above equation can also be written as

$$\pm e^{\mp |\nu| \vartheta_\nu} = \frac{\tan \varphi_1}{\cosh^2 (|\varkappa_\nu| |a_1|) (\tan \alpha_\nu \pm \tan \varphi_1)}$$

or

$$\mp |\nu| \vartheta_\nu = \ln \left[\frac{1}{\cosh^2 (|\varkappa_\nu| |a_1|)} \cdot \left| \frac{\tan \varphi_1}{\tan \alpha_\nu \pm \tan \varphi_1} \right| \right]$$

From the first equation in (7) we obtain

$$r_1 - r_\nu = C_\nu$$

or from eqs. (202)

$$r_1 - i |a_1| = r_\nu \mp |\varkappa_\nu| |a_1|$$

or, using the first equation in (97),

$$\tan (r_\nu \mp |\varkappa_\nu| |a_1|) = \tanh |a_1| \tan \varphi_1$$

where the upper sign corresponds to the upper sign in eqs. (202) and vice versa.

If the lower sign is valid, we rotate the coordinate axis through 180° by changing the sign for r_ν. The last equation can then be written as

$$\tan (r_\nu - |\varkappa_\nu| |a_1|) = \pm \tanh |a_1| \tan \varphi_1$$

We now introduce a new parameter determined by

$$\left.\begin{aligned} &\varphi_\nu = \varphi_1 \text{ for the angle between the centres of rotation } C_\nu = |a_1| (i - |\varkappa_\nu|) \\ &\varphi_\nu = -\varphi_1 \text{ for the angle between the centres of rotation } C_\nu = |a_1| (i + |\varkappa_\nu|) \end{aligned}\right\} \quad (204)$$

The above equations for the rolling curve in parametric form then become

$$\left.\begin{aligned} &\tan (r_\nu - a_\nu) = \frac{\tan a_\nu}{\tan \alpha_\nu} \tan \varphi_\nu = \tanh |a_1| \tan \varphi_\nu \\ &\pm e^{\mp |\nu| \vartheta_\nu} = \frac{\tan \varphi_\nu}{\cos^2 a_\nu (\tan \alpha_\nu + \tan \varphi_\nu)} \text{ or } \mp |\nu| \vartheta_\nu = \ln \left[\frac{1}{\cos^2 a_\nu} \cdot \left| \frac{\tan \varphi_\nu}{\tan \alpha_\nu + \tan \varphi_\nu} \right| \right] \end{aligned}\right\} \quad (205)$$

where

$$\left.\begin{aligned} \tan \alpha_\nu &= \frac{\tan a_\nu}{\tanh |a_1|} \\ |\nu| &= \frac{\sin a_\nu}{\sinh |a_1|} \\ a_\nu &= |\varkappa_\nu| |a_1| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (206)$$

ϑ_ν is measured in radians.

If we put $\varkappa_\nu = 1$, i.e. $|\varkappa_\nu| = i\,\varkappa_\nu = i$, we return to the basic curve, eqs. (97) (apart from the $\pm$ signs before e, which in this case only implies a rotation $= \pi$ radians of the polar axis). In comparison with the basic curve, it can be seen that the upper sign before $|\nu|\vartheta_\nu$ corresponds to the A-branches and the lower sign corresponds to the B-branches. The whole curve is obtained even if φ_ν varies only over 180°. As for the basic curve, however, the range between 0 and 180° is reserved for the A-branches, for which the negative sign before $|\nu|\vartheta_\nu$ applies, and the range between 180° and 360° for the B-branches, for which the positive sign before $|\nu|\vartheta_\nu$ applies.

As the main curve we define the curve for which $r_\nu - |\varkappa_\nu||a_1|$ lies in the same quadrant as φ_ν. Thus, $r_\nu - |\varkappa_\nu||a_1|$ lies for the antipode curve in the same quadrant as $180° + \varphi_\nu$, similar to the conditions for the basic curve.

For reasons which will be discussed later on, the branches obtained are called tan-branches, if

$$-\alpha_\nu < \varphi_\nu < 180° - \alpha_\nu$$

and cot-branches, if

$$180° - \alpha_\nu < \varphi_\nu < 360° - \alpha_\nu$$

The tan-branch which belongs to the A-curve is called tan-branch A, and the one belonging to the B-curve is called tan-branch B. The cot-branch which belongs to the A-curve is called cot-branch A, and the one belonging to the B-curve is called cot-branch B (see Figs. 10 a and b).

The equation in non-parametric form is obtained by eliminating φ_ν from eqs. (205). We obtain after simplifying

$$\tan r_\nu = \frac{\tan a_\nu}{1 \mp e^{\mp|\nu|\vartheta_\nu}} = \frac{\tanh |a_1| \tan \alpha_\nu}{1 \mp e^{\mp|\nu|\vartheta_\nu}} \quad \ldots\ldots\ldots\ldots\ldots \quad (207)$$

As far as the significance of the signs in eqs. (205) and (207) is concerned, the following conditions apply for both equations:
The upper sign in $\mp |\nu|\vartheta_\nu$ gives the A-branches for which we have $0 < \varphi_\nu < 180°$
The lower sign in $\mp |\nu|\vartheta_\nu$ gives the B-branches for which we have $-180° < \varphi_\nu < 0$
The upper sign before e corresponds to

$$\frac{\tan \varphi_\nu}{\frac{\tan a_\nu}{\tanh |a_1|} + \tan \varphi_\nu} > 0 \text{ or } \frac{\tan \varphi_\nu}{\tan \alpha_\nu + \tan \varphi_\nu} > 0 \text{ or } \begin{cases} 0 < \varphi_\nu < 180° - \alpha_\nu \\ 180° < \varphi_\nu < -\alpha_\nu \end{cases}$$

i.e. to the tan-branch A or cot-branch B,
and the lower sign before e corresponds to

$$\frac{\tan \varphi_\nu}{\frac{\tan a_\nu}{\tanh |a_1|} + \tan \varphi_\nu} < 0 \text{ or } \frac{\tan \varphi_\nu}{\tan \alpha_\nu + \tan \varphi_\nu} < 0 \text{ or } \begin{cases} -\alpha_\nu < \varphi_\nu < 0 \\ 180° - \alpha_\nu < \varphi_\nu < 180° \end{cases}$$

i.e. to the tan-branch B or cot-branch A.

As the basic curve is the limiting case for the imaginary ellipse and the imaginary hyperbola if $e \to 0$ and $\vartheta_1 \to \pm\infty$, we can also obtain the above rolling curves as the limiting case for the rolling curves to the imaginary conic sections when $e \to 0$ and $\vartheta_\nu \to \pm\infty$.

If we make such a derivation by starting from the rolling curves to the imaginary ellipse, it is found that the tan-branches for the imaginary ellipse correspond to the tan-branches of the rolling curve treated above, and similarly for the cot-branches. This explains why the terms tan-branches and cot-branches have also been used for the curves discussed above.

In order to give an idea of the shape of the curves, a rolling curve having $a_1 = 27.566°$, $\alpha_1 = 30°$, $a_\nu = |\varkappa_\nu||a_1| = 30°$ and $|\nu| = 1$ has been computed, and is shown in Figs. 10 a and 10 b. Fig. 10 a shows the main curves of the tan-branches and Fig. 10 b the main curves of the cot-branches.

If in eq. (207) we substitute

$$k_1 = \tan a_\nu$$

the equation can be written as

$$\left.\begin{aligned} &\tan r_\nu = \frac{k_1}{1 \mp e^{\mp|\nu|\vartheta_\nu}} \quad (k_1 \text{ is positive}) \\ &\tan a_\nu = k_1 \\ &\tan \alpha_\nu = \frac{k_1}{\tanh\ |a_1|} \\ &\alpha_1 = 0 \\ &\sinh\ |a_1| = \frac{\sin\ a_\nu}{|\nu|} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (208)$$

This system of equations is used if we have a curve given in the same form as the first equation in the system of equations. By using the following equations, it is then possible to determine $a_\nu = |\varkappa_\nu||a_1|$, $|a_1|$, $|\nu|$ and $|\varkappa_\nu|$, after which the equation may be expressed in the forms (205) or (207). The system of equations always gives real solutions.

2) *Special cases of the rolling curves treated under D*

α) *The crown-wheel curve (the spherical exponential curve)*

If in eqs. (205) and (206) we put $a_\nu = |\varkappa_\nu||a_1| = 90°$, we obtain

$$\left.\begin{aligned} &\tan (r_\nu - 90°) = \tanh\ |a_1|\ \tan \varphi_\nu \\ &\pm e^{\mp|\nu|\vartheta_\nu} = \frac{1}{\cos\ a_\nu} \cdot \frac{\tan \varphi_\nu}{\dfrac{\sin\ a_\nu}{\tanh\ |a_1|} + \cos a_\nu \tan \varphi_\nu} \\ &\alpha_\nu = 90° \\ &|\nu| = \frac{1}{\sinh\ |a_1|} \end{aligned}\right\}$$

Since, however, $a_\nu \to 90°$, the second equation may also be written as

$$\pm e^{\mp|\nu|\vartheta_\nu} = \frac{1}{\cos\ a_\nu} \cdot \frac{\tan \varphi_\nu}{\dfrac{\sin\ a_\nu}{\tanh\ |a_1|} + \cos a_\nu \tan \varphi_\nu} \to \frac{\tanh\ |a_1|\ \tan \varphi_\nu}{\sin a_\nu \cos a_\nu}$$

or

$$\pm (\sin a_\nu \cos a_\nu)\ e^{\mp|\nu|\vartheta_\nu} = \tanh\ |a_1|\ \tan \varphi_\nu$$

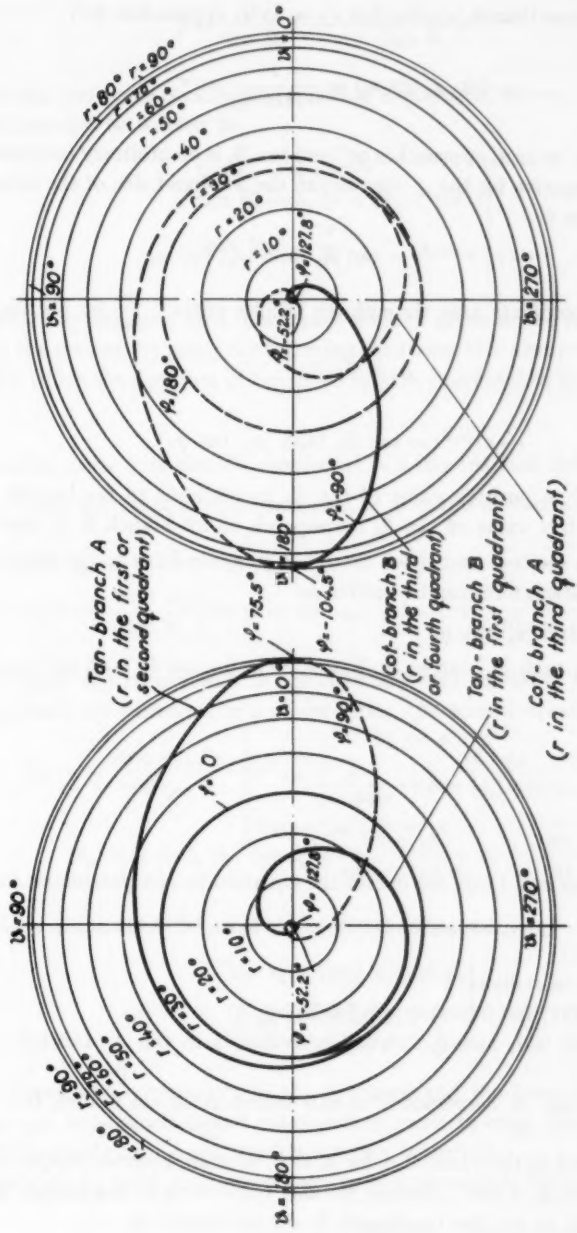


Fig. 10. Rolling curves (main curves) to the imaginary spherical conic section when $\varepsilon \to 0$, both with $\alpha_1 = 30°$, $a_p = |\varkappa_p|/|\varkappa_1| = 30°$ and $|p| = 1$.

a) Tan-branches b) Cot-branches

From this it can be seen that ϑ_ν approaches $\pm\infty$ as a_ν approaches 90°.

Further, we put

$$\pm (\sin a_\nu \cos a_\nu) = \frac{1}{-\tan R_s} \cdot e^{-w}$$

where w approaches $+\infty$ as a_ν approaches 90° and $\tan R_s$ is an arbitrary constant, positive for the — sign and negative for the + sign before the left-hand side of the equation. The equation then changes to

$$-e^{-w \mp |\nu| \vartheta_\nu} = \tan R_s \tanh |a_1| \tan \varphi_\nu$$

We now rotate the coordinate axes through the infinite angle $\pm \frac{w}{|\nu|}$ by putting

$$\vartheta_\nu = \bar{\vartheta}_\nu \mp \frac{w}{|\nu|}$$

We then obtain

$$-e^{\mp |\nu| \bar{\vartheta}_\nu} = \tan R_s \tanh |a_1| \tan \varphi_\nu$$

where, since $\alpha_\nu = 90°$, a positive value of $\tan R_s$ corresponds to tan-branch B or cot-branch A and a negative value of $\tan R_s$ corresponds to cot-branch B or tan-branch A. If the bars on $\bar{\vartheta}_\nu$ are omitted and if we choose $R_s = r_\nu$ for $\vartheta_\nu = 0$, the equation of the rolling curve in parametric form can be written as

$$\left.\begin{aligned} &\tan (r_\nu - 90°) = \tanh |a_1| \tan \varphi_\nu \\ &e^{\mp |\nu| \vartheta_\nu} = -\tan R_s \tanh |a_1| \tan \varphi_\nu \text{ or } \mp |\nu| \vartheta_\nu = \ln (-\tan R_s \tanh |a_1| \tan \varphi_\nu) \end{aligned}\right\} \quad (209)$$

where

$$\left.\begin{aligned} &\alpha_\nu = 90° \\ &|\nu| = \frac{1}{\sinh |a_1|} \\ &a_\nu = |\varkappa_\nu| \, |a_1| = 90° \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (210)$$

By eliminating φ_ν from eqs. (209), we obtain the equation in non-parametric form

$$\tan r_\nu = \tan R_s \, e^{\pm |\nu| \vartheta_\nu} = \tan R_s \, e^{\pm \vartheta_\nu / \sinh |a_1|} \quad \ldots\ldots\ldots\ldots \quad (211)$$

where ϑ_ν is measured in radians.

In eqs. (209) and (211) the following conditions apply:

The upper sign before $|\nu| \vartheta_\nu$ and $\vartheta_\nu / \sinh |a_1|$ corresponds to the A-branches
" lower " " " " " " " " " B- "
A negative value of $\tan R_s$ corresponds to tan-branch A or cot-branch B
" positive " " " R_s " " " B " " A

R_s may be considered as the value of r_ν for $\vartheta_\nu = 0$. We can, however, obtain all types of curve by choosing $0 < R_s < 180°$, because for this type of curve tan-branch A and cot-branch A are identical, as are also tan-branch B and cot-branch B.

If we project radially the curve on to a cylinder, tangential to the great circle $r = 90°$, and if we place the x-axis on the great circle, and the y-axis along a generatrix, then because $y = \tan (90° - r_\nu)$ the equation of the projected curve can be written as

$$y = \frac{1}{\tan R_s} e^{\mp|\nu|\vartheta_\nu} = \frac{1}{\tan R_s} e^{\mp|\nu|x}$$

Because of this, the curve is called the spherical exponential curve.

Eq. (211) can also be written as

$$\left.\begin{aligned} &\tan r_\nu = k_1 \, e^{\pm|\nu|\vartheta_\nu} \\ &\tan R_s = k_1 \; (0 < R_s < 180°) \\ &\sinh |a_1| = \frac{1}{|\nu|} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (212)$$

This system of equations is used if we have a curve given in the same form as the first equation in the system. By using the following equations, it is always possible to determine $\tan R_s$, after which the equation of the curve may be expressed as in (209) or (211).

β) The rolling curve with periodic number $|\nu| = 0$ *(the spherical helix)*

If in eqs. (205) and (206) $a_\nu \to 0$, then also $|\nu| \to 0$ and $\alpha_\nu \to 0$, and the equations take the form

$$\left.\begin{aligned} &\tan r_0 = \tanh |a_1| \tan \varphi_0 \\ &\vartheta_0 = \pm \frac{\sinh |a_1|}{\sin a_0} \left[\ln \, (1 - \sin^2 a_0) + \ln \left| 1 + \frac{\tan a_0}{\tanh |a_1| \tan \varphi_0} \right| \right] \end{aligned}\right\}$$

where the upper sign corresponds to the A-branch and the lower sign to the B-branch. If the logarithms are expanded in a power series, the second equation can be written as

$$\vartheta_0 \to \pm \frac{\sinh |a_1|}{\sin a_0} \left(- \sin^2 a_0 + \ldots + \frac{\tan a_0}{\tanh |a_1| \tan \varphi_0} - \ldots \right)$$

or

$$\vartheta_0 \to \mp \sinh |a_1| \sin a_0 \pm \frac{\cosh |a_1|}{\cos a_0 \tan \varphi_0} \to \pm \frac{\cosh |a_1|}{\tan \varphi_0}$$

The equations of the curve in parametric form thus become

$$\left.\begin{aligned} &\tan r_0 = \tanh |a_1| \tan \varphi_0 \\ &\pm \vartheta_0 = \frac{\cosh |a_1|}{\tan \varphi_0} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (213)$$

where the upper sign corresponds to the A-branch and the lower sign to the B-branch.

Since $a_0 \to 0$, tan-branch B and cot-branch A cease to exist. The upper sign therefore corresponds to tan-branch A and the lower to cot-branch B.

By eliminating φ_0 from (213), we obtain, in non-parametric form, the equation

$$\tan r_0 = \frac{\sinh |a_1|}{\pm \vartheta_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (214)$$

where the same laws for signs are valid.

If the curve is projected radially from the centre of the sphere on to a cylinder tangential to the great circle $r = 90°$, and if we place the x-axis on the great circle and the y-axis along a generatrix, the equation of the projected curve can be written as

$$y = \frac{\pm \vartheta_0}{\sinh |a_1|} = \frac{\pm x}{\sinh |a_1|}$$

The radial projection on the cylinder is therefore a helix and, because of this, the curve is called the spherical helix.

Eq. (214) can also be written as

$$\left.\begin{aligned} \tan r_0 &= \frac{k_1}{\pm \vartheta_0} \ (k_1 \text{ is positive}) \\ \sinh |a_1| &= k_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (215)$$

This system of equations is used if we have a curve given in the same form as the first equation in the system. By using the second equation, it is always possible to determine $\sinh |a_1|$, after which the equation of the curve may be expressed as in (213) or (214).

b) The combination of the rolling curves treated under D to form rolling curve pairs and the angles between the centres of rotation involved

From the derivations of the equations of the rolling curves, it can be seen that all rolling curves having the same value of $|a_1|$ and α_1 as the basic curve can form rolling curve pairs with the basic curve with two different angles between the centres of rotation according to eqs. (202), in which case the relations between φ_r and φ_1 are given by eq. (204). Both curves are then either main curves or antipode curves. Applying the same reasoning as was used for curves of other types, i.e., if we imagine two rolling curves rolling simultaneously on the same basic curve in such a way that the point of contact is common for all three curves, we easily find that two rolling curves which are both main curves or both antipode curves, and which have the same $|a_1|$- and α_1-values, can form rolling curve pairs with each other with two different angles between their centres of rotation, namely

$$\left.\begin{aligned} C_{L-R} &= a_{rL} - a_{rR} = |a_1| \ (|\varkappa_L| - |\varkappa_R|) \text{ in which case } \varphi_R = \varphi_L \\ C_{L-R} &= a_{rL} + a_{rR} = |a_1| \ (|\varkappa_L| + |\varkappa_R|) \text{ in which case } \varphi_R = -\varphi_L \end{aligned}\right\} \quad \ldots. \quad (216)$$

By replacing the centre of rotation by its antipode, further rolling possibilities can be obtained. For reasons of symmetry, it is easy to see that replacing the centre of rotation by its antipode is the same as replacing tan-branch A by cot-branch B, and tan-branch B by cot-branch A or vice versa, in which case φ_r is replaced at the same time by $\varphi_r \pm 180°$ and C_{L-R} by $C_{L-R} \mp 180°$. We then obtain the following different possible combinations of rolling curves, if the angle between the centres of rotation is measured by the shorter arc around the great circle.

Rolling case 1

$$\left.\begin{aligned} &\varphi_R = \varphi_L \\ &C_{L-R} = a_{rL} - a_{rR} = |a_1| \ (|\varkappa_L| - |\varkappa_R|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (217)$$

Rolling case 2

$$\left.\begin{aligned} \varphi_R &= -\varphi_L \\ C_{L-R} &= a_{\nu L} + a_{\nu R} = |a_1|\,(|\varkappa_L| + |\varkappa_R|) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (218)$$

Rolling case 3

$$\left.\begin{aligned} \varphi_R &= 180^\circ - \varphi_L \\ C_{L-R} &= a_{\nu L} + a_{\nu R} - 180^\circ = |a_1|\,(|\varkappa_L| + |\varkappa_R|) - 180^\circ \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (219)$$

Rolling case 4

$$\left.\begin{aligned} \varphi_R &= 180^\circ + \varphi_L \\ C_{L-R} &= a_{\nu L} - a_{\nu R} \mp 180^\circ = |a_1|\,(|\varkappa_L| - |\varkappa_R|) \mp 180^\circ \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (220)$$

where the upper sign is valid if $a_{\nu L} > a_{\nu R}$ and vice versa.

For all four rolling cases both curves in the rolling curve pair must be either main curves or antipode curves.

In the same way as for rolling curves to the other spherical conic sections, we can find that the last four equations give one, but only one, solution for $a_\nu = |\varkappa_\nu||a_1| \leq 90^\circ$. Therefore, for a given value of C_{L-R} we have one, but only one, mating curve to a given rolling curve.

To obtain a compilation of the combinations of rolling curves which give external meshing with the absolute value of the angle between the centres of rotation $< 180^\circ$, we can proceed in the same way as for the other types of rolling curve, i.e. we investigate the nature of meshing if a main or antipode rolling curve rolls on another main or antipode rolling curve. The nature of meshing can only be altered, and is always altered, when the radius vector of one of the rolling curves changes its sign, i.e., when a tan-branch changes to a cot-branch or vice versa. It is, therefore, easy to determine the nature of meshing during the rolling if we imagine the curves rolling on each other from $\varphi_L = 0$ to $\varphi_L = 360^\circ$, and if we notice that, e.g. two main curve tan-branches A roll against each other with internal meshing if $\varphi_R = \varphi_L$, and that a tan-branch A and a tan-branch B roll against each other with external meshing if $\varphi_R = -\varphi_L$. In the cases where we obtain, during rolling, internal meshing, the internally meshed curve is replaced as stated above, and at the same time φ_ν is also replaced by $\varphi_\nu \pm 180^\circ$ and the angle between the centres of rotation is changed by $\pm 180^\circ$. In this way, we can obtain all the possible combinations giving external meshing and having $|C_{L-R}| < 180^\circ$, which are tabulated below.

The rolling curve having the greatest a_ν- and α_ν-values	The rolling curve having the smallest a_ν- and α_ν-values	Rolling case	At point of contact $\varphi_R =$
tan-branch B of main curve cot-branch A of antipode curve	cot-branch B of main curve tan-branch A of antipode curve	1	φ_L
tan-branch B of main curve cot-branch A of antipode curve tan-branch A of main curve cot-branch B of antipode curve	tan-branch A of main curve cot-branch B of antipode curve tan-branch B of main curve cot-branch A of antipode curve	2	$-\varphi_L$
cot-branch B of main curve tan-branch A of antipode curve	cot-branch B of main curve tan-branch A of antipode curve	3	$180^\circ - \varphi_L$
cot-branch B of main curve tan-branch A of antipode curve cot-branch A of main curve tan-branch B of antipode curve	tan-branch A of main curve cot-branch B of antipode curve tan-branch B of main curve cot-branch A of antipode curve	4	$180^\circ + \varphi_L$

Appendix 3 gives a compilation of the relation between φ_L and φ_R and also the angle between the centres of rotation for spherical conic section rolling curves of all types, when the curves are rolling with external meshing and when $|C_{L-R}| < 180°$. For rolling curves of all types we have four different rolling cases.

If we always choose $-180° < r_\nu < +180°$ and if we always measure C_{L-R} as the minor arc of the great circle through the centres of rotation, we obtain for external meshing for spherical conic section rolling curves of all types

$$|C_{L-R}| = |r_L| + |r_R| \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (221)$$

E. Relation between the angles of displacement for the curves (or wheels) in a rolling curve pair (or gear) consisting of spherical conic section rolling curves

When designing non-circular bevel gears, it is often necessary to calculate the angle of displacement of the one wheel as a function of the angle of displacement of the other. An expression for the relation between the polar angles ϑ_R and ϑ_L to the point of contact of the rolling curves must first be derived. The equations previously derived can be used directly for the derivation, particularly the second equation in the system of equations in parametric form with the parameter angle φ_ν. Of the two alternative expressions for the imaginary spherical ellipse, eqs. (144), we use the expression

$$\operatorname{amh} |\nu| \vartheta_\nu = \frac{\alpha_\nu - \varphi_\nu}{180° - (\alpha_\nu - \varphi_\nu)}$$

For spherical conic section rolling curves of all types we have, according to Appendix 3, four different cases of rolling, namely

1) $\varphi_R = \varphi_L$
2) $\varphi_R = -\varphi_L$
3) $\varphi_R = 180° - \varphi_L$
4) $\varphi_R = 180° + \varphi_L$

By using that relation which is applicable we eliminate φ_R from the equation for the R-curve and then determine:
For rolling curves to the real conic section and to the imaginary spherical hyperbola:

$$\tan \frac{90° - \varphi_L}{2}$$

For rolling curves to the imaginary spherical ellipse:

$$\varphi_L - 90°$$

For rolling curves to the limiting curve to the imaginary spherical conic sections when $e \to 0$ and $\vartheta \to \infty$:

$$\frac{1}{\tan \varphi_L}$$

The expression thus obtained for the R-curve must be equal to the corresponding expression for the L-curve, which gives the desired relation between ϑ_R and ϑ_L. The result obtained for the different types of spherical rolling curve when using this method are tabulated in Appendix 6.

The equations in Appendix 6 give the relation between ϑ_R and ϑ_L. For the calculation of the relation between the angles of displacement γ_R and γ_L of the wheels from a given initial position, the angle ϑ for the one curve, corresponding to the initial position, e.g. $\vartheta_L = (\vartheta_L)_s$, is first substituted in the equations, after which the corresponding angle for the other curve, $\vartheta_R = (\vartheta_R)_s$, is calculated. Thereafter, $\vartheta_R = \gamma_R + (\vartheta_R)_s$ and $\vartheta_L = \gamma_L + (\vartheta_L)_s$ are substituted in the original equation, which then gives the relation between the angles of displacement γ_R and γ_L of the wheels, measured from a certain initial position. It is often convenient to choose this initial position at ϑ_L or $\vartheta_R = 0$, but this is not always possible. In such cases we choose the initial position at the end of the arc used. With internal meshing, γ_R and γ_L have equal signs, and with external meshing opposite signs. For bevel gears, however, where we always have external meshing, γ_R and γ_L always have opposite signs.

F. Concerning ratio curves

When computing non-circular bevel gears, it is not generally necessary to know the equations of the ratio curves for the spherical conic section rolling curves. If, at any time, however, an expression for the gear ratio is desired, this can be calculated in a way similar to that shown below.

The equation giving the relation between ϑ_R and ϑ_L taken from Appendix 6 for the rolling curves in question is differentiated. We then introduce

$$u_{L/R} = \frac{d\vartheta_L}{d\vartheta_R} \qquad (222)$$

Finally, ϑ_R is eliminated from the two equations and we then obtain the equation for the gear ratio curve.

The method will be illustrated by an example. We assume that both the L-curve and the R-curve are rolling curves with a real ν-value to the spherical ellipse. According to Appendix 6 we then have

$$\begin{matrix}\tan \\ \cot\end{matrix} \nu_R \frac{\vartheta_R}{2} \tan \frac{90^\circ \pm \alpha_{\nu R}}{2} = \pm \tan \nu_L \frac{\vartheta_L}{2} \tan \frac{90^\circ + \alpha_{\nu L}}{2}$$

and the rules for signs and for $\begin{matrix}\tan \\ \cot\end{matrix}$ for the four different rolling cases are also given in Appendix 6.

We now introduce the substitution

$$\tan\frac{90^\circ+\alpha_{L+R}}{2}=\tan\frac{90^\circ\mp\alpha_{\nu R}}{2}\tan\frac{90^\circ+\alpha_{\nu L}}{2} \quad \ldots\ldots\ldots\ldots (223)$$

where the upper sign is valid for rolling cases 1 and 3
" lower " " " " " " " 2 " 4

For all four rolling cases the equation can now be written as

$$\begin{matrix}\tan\\ \cot\end{matrix}\nu_R\frac{\vartheta_R}{2}=\pm\tan\frac{90^\circ+\alpha_{L+R}}{2}\tan\nu_L\frac{\vartheta_L}{2} \quad \ldots\ldots\ldots\ldots (224)$$

where, in the left-hand side of the equation, the upper expression is valid for rolling cases 1 and 3, and the lower expression for rolling cases 2 and 4. In the right-hand side of the equation the upper sign is valid for rolling cases 1 and 2 and the lower sign for rolling cases 3 and 4.

By differentiating this equation, substituting $u_{L/R}=\frac{d\vartheta_L}{d\vartheta_R}$, and solving with respect to $\frac{1}{u_{L/R}}$, we obtain

$$\frac{1}{u_{L/R}}=\pm\frac{\nu_L}{\nu_R}\tan\frac{90^\circ+\alpha_{L+R}}{2}\cdot\frac{1}{\pm\left[1+\begin{matrix}\tan^2\\ \cot^2\end{matrix}\nu_R\frac{\vartheta_R}{2}\right]\cos^2\nu_L\frac{\vartheta_L}{2}}$$

Inserting $\tan^2\nu_R\frac{\vartheta_R}{2}$ or $\cot^2\nu_R\frac{\vartheta_R}{2}$ according to eq. (224) gives, after simplification,

$$\frac{1}{u_{L/R}}=\pm\frac{\nu_L}{\nu_R}\cdot\frac{\cos\alpha_{L+R}}{1-\sin\alpha_{L+R}\cos\nu_L\vartheta_L} \quad \ldots\ldots\ldots\ldots (225)$$

where the upper sign before the right-hand side of the equation corresponds to rolling cases 1 and 4, giving internal meshing, and the lower sign to rolling cases 2 and 3, giving external or double internal meshing.

If $a_L=a_R$, thus also $\alpha_L=\alpha_R$; then for rolling cases 2 and 4 eq. (223) becomes

$$\tan\frac{90^\circ+\alpha_{L+R}}{2}=\frac{1+\tan\frac{\alpha_{L+R}}{2}}{1-\tan\frac{\alpha_{L+R}}{2}}=\tan^2\frac{90^\circ+\alpha_L}{2}=\frac{1+\sin\alpha_L}{1-\sin\alpha_L}$$

from which

$$\tan\frac{\alpha_{L+R}}{2}=\sin\alpha_L$$

For these special cases the angle α_{L+R} is thus the same angle as α_T used in eq. (110).

It should be noted that if $\frac{1}{u_{L/R}}$ is plotted in polar coordinates, then this curve will be a plane conic section rolling curve. This is generally valid for spherical conic section rolling curves of all types.

G. The angle between the radius vector and the normal for spherical conic section rolling curves

Because all spherical conic section rolling curves can roll without slipping on the corresponding basic curves (spherical conic section curves) having the same a_1- and α_1-values, the angle Φ at the point of contact is the same, both for the basic curve and the rolling curve. As, moreover, the rolling curve can roll on the basic curve in such a manner that φ for the two curves has the same value, it can be seen that the previously deduced expressions (58) or (59), (74) or (75), (89) or (90) and (98) or (99) apply also to the rolling curves.

If we have the same φ-value for the basic curve and for the rolling curve, the following rules are valid for the different types of rolling curve, which can be seen from the equations in parametric form:

For rolling curves to the real spherical conic section where the basic curve is obtained if we put $\varkappa_\nu = 1$

$$r - a = r_\nu - \varkappa_\nu a_1 = r_\nu - a_\nu$$

For rolling curves to the imaginary spherical ellipse where the basic curve is obtained if $|\varkappa_\nu| = i$

$$r - i\,|a| = r_\nu - |\varkappa_\nu||a_1| = r_\nu - a_\nu$$

For rolling curves to the imaginary spherical hyperbola where the basic curve is obtained if we put $\varkappa_\nu = i\,|\varkappa_\nu| = 1$, i.e. $|\varkappa_\nu| = -i$

$$r - (90^\circ + i\,|a|) = r_\nu - (90^\circ - |\varkappa_\nu||a_1|) = r_\nu - \bar{a}_\nu$$

For rolling curves to the limiting curve for the imaginary spherical conic sections when $e \to 0$ and where the basic curve is obtained if we put $|\varkappa_\nu| = i$

$$r - i\,|a| = r_\nu - |\varkappa_\nu||a_1| = r_\nu - a_\nu$$

where in all cases

r = the radius vector of the basic curve

r_ν = the radius vector of the rolling curve

The equations are valid assuming that the rolling curves are both either main curves or antipode curves. With the aid of these equations, the previously derived expressions for $\tan \Phi$ can be transformed into the following expressions, where suffix 1 indicates values belonging to the basic curve.

For rolling curves to the real spherical conic section we have from eqs. (58) and (59)

$$\tan \Phi = \pm \frac{\tan \alpha_1 \cos \varphi}{\cos a_1 \sqrt{1 + \tan^2 a_1 \sin^2 \alpha_1 \sin^2 \varphi}} = \frac{\tan \alpha_1 \cos(r_\nu - a_\nu)\cos \varphi}{\cos a_1} \quad \ldots(226)$$

Eq. (226) becomes indeterminate if $a_1 = 90^\circ$ and $\alpha_1 = 0$. According to the third equation in (52) we then introduce

$$\tan a_1 \sin \alpha_1 = \tan e a_1$$

and obtain

$$\tan \Phi = \pm \frac{\tan e a_1 \cos \varphi}{\sqrt{1 + \tan^2 e a_1 \sin^2 \varphi}} = \tan e a_1 \cos (r_\nu - a_\nu) \cos \varphi \quad \ldots\ldots \quad (227)$$

For rolling curves to the imaginary spherical ellipse we obtain from eqs. (74) and (75)

$$\tan \Phi = \pm \frac{\tan \alpha_1}{\cosh |a_1| \sqrt{1 - (1 - \tanh^2 |a_1| \sin^2 \alpha_1) \sin^2 \varphi}} = \frac{\tan \alpha_1 \cos (r_\nu - a_\nu)}{\cosh |a_1| \cos \varphi} \quad (228)$$

In the special case where $\alpha_1 = 90°$ and $|a_1| = \infty$, eq. (228) becomes indeterminate. According to eq. (159), however, we then obtain

$$\lim \frac{\tan \alpha_1}{\cosh |a_1|} = |\nu|$$

If this expression and also $\alpha_1 = 90°$ and $|a_1| = \infty$ are inserted in eq. (228), we obtain for this special case

$$\tan \Phi = \pm |\nu| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (229)$$

For rolling curves to the imaginary spherical hyperbola we obtain from eqs. (89) and (90)

$$\tan \Phi = \pm \frac{\sin \alpha_1 \cos \varphi}{\sinh |a_1| \sqrt{1 + \frac{\tan^2 \alpha_1}{\tanh^2 |a_1|} \sin^2 \varphi}} = \frac{\sin \alpha_1 \cos (r_\nu - \bar{a}_\nu) \cos \varphi}{\sinh |a_1|} \quad \ldots\ldots \quad (230)$$

In the special case when $\alpha_1 = 0$ and $|a_1| = 0$, eq. (230) becomes indeterminate, but, as stated on page 85, this special case may also be considered as a rolling curve to the real spherical conic section, and therefore need not be considered here.

For rolling curves to the limiting curve for the imaginary conic sections when $e \to 0$, we obtain from eqs. (98) and (99)

$$\tan \Phi = \pm \frac{1}{\sqrt{\frac{\cosh^2 |a_1|}{\sin^2 \varphi} - 1}} = \pm \frac{|\sin (r_\nu - a_\nu)|}{\sinh |a_1|} \quad \ldots\ldots\ldots\ldots \quad (231)$$

In all cases where we have double signs the upper sign is valid for the main curves and the lower sign for the antipode curves.

Appendix 5 gives a compilation of the formulae derived above for evaluating the angle Φ.

H. The arc length of spherical conic section rolling curves

Because a rolling curve can roll without slipping on the basic curve in such a way that the point of contact corresponds to the same value of φ for the two curves, the previously deduced expressions for the arc length of the basic curve are also valid for the rolling curves. A few limiting cases must, however, be treated specially.

For the real spherical conic section and its rolling curves eqs. (60), (64) and (67) are valid.

If, however, $\sin \alpha_1 = 0$ and $a_1 = 90°$, then these expressions become indeterminate. The limiting value can, however, be obtained by using the third equation in (52) and the first equation in (108), giving

$$\tan a_1 \sin \alpha_1 = \tan a_\nu \sin \alpha_\nu = \tan e a_1$$

If the equation of the curve is known, $\tan a_1 \sin \alpha_1 = \tan ea_1$ must also be known. Eq. (60) now becomes

$$dS_e = \sqrt{\tan^2 a_1 \sin^2 \alpha_1 + \cos^2 \alpha_1 \sin^2 a_1} \cdot \frac{d\varphi}{1 + \tan^2 ea_1 \sin^2 \varphi}$$

If $a_1 = 90°$ and $\alpha_1 = 0$ this expression changes to

$$dS_e = \frac{1}{\cos ea_1} \cdot \frac{d\varphi}{1 + \tan^2 ea_1 \sin^2 \varphi}$$

or, after integration,

$$S_e = \arctan \frac{\tan \varphi}{\cos ea_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (232)$$

where S_e is measured from the point where $\varphi = 0$.

It can be shown that the main curve corresponding to this special case having $a_1 = 90°$ and $\alpha_1 = 0$ is a great circle with the origin (one of the foci) at an angular distance of ea_1 from the centre of the circle.

The arc length for the imaginary spherical ellipse and its rolling curves is given by eqs. (76), (79) and (82). If for rolling curves to the imaginary spherical ellipse $\sin \alpha_1 \tanh |a_1| = 1$, then eq. (76) will give an indeterminate value of dS. This means that both $|a_1| = \infty$ and $\alpha_1 = 90°$, which corresponds to the special case B a 2 γ in Chapter V, p. 72. Eq. (76) then changes to

$$dS_{ei} = \sinh |a_1| \sqrt{1 - \sin^2 \alpha_1 \tanh^2 |a_1|}\, d\varphi$$

or

$$dS_{ei} = \tanh |a_1| \sin \alpha_1 \sqrt{\frac{\cosh^2 |a_1|}{\tan^2 \alpha_1} + 1}\, d\varphi$$

or, since $\tanh |a_1| \sin \alpha_1 = 1$,

$$dS_{ei} = \sqrt{\frac{\cosh^2 |a_1|}{\tan^2 \alpha_1} + 1}\, d\varphi$$

According to the second equation in (159) we have, however,

$$\lim \frac{\tan \alpha_1}{\cosh |a_1|} \to |\nu|$$

which inserted in the above expression gives, after integration,

$$S_{ei} = \sqrt{1 + \frac{1}{|\nu|^2}}\, \varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (233)$$

where S_{ei} is measured from the point where $\varphi = 0$.

For the imaginary spherical ellipse we cannot have at the same time $|a_1| = \infty$ and $\alpha_1 \neq 90°$. This can be understood from the fact that we should then obtain $dS = \infty$.

The arc length for the imaginary spherical hyperbola according to eq. (91) becomes indeterminate if $|a_1| = 0$ and $\alpha_1 = 0$. This, however, corresponds to the same curve as the real spherical conic section having $a_1 = 90°$ and $\alpha_1 = 0$, as, if we choose $\tan ea_1 = \frac{1}{\tan |e| |a_1|}$, the equations for both curves then become

$$\left.\begin{aligned} \tan\,(\nu - 90°) &= \tan\, e a_1\, \sin\,\varphi = \frac{1}{\tan\,|e|\,|a_1|}\sin\varphi \\ \tan\frac{\vartheta}{2} &= \tan\frac{90° - \varphi}{2} \end{aligned}\right\}$$

The case when $|a_1| = 0$ and $\alpha_1 = 0$, which is thus a limiting case between the real spherical conic section and the imaginary spherical hyperbola, therefore need not be treated separately, as it can be included among the real spherical conic sections. Eqs. (91), (94) and (95) are thus always valid for the imaginary spherical hyperbola and its rolling curves.

According to eq. (100) the arc length for the limiting curve to the imaginary spherical conic sections when $e \to 0$ never assumes an indeterminate form.

For the different types of curve, we find directly from the expression for the arc length and from the equations of the curves the following rules:

For rolling curves to the real spherical conic section:

By increasing ϑ {For curves with real values of ν and the larger branch of curves with imaginary values of ν: φ and S decrease. For the smaller branch of curves with imaginary values of ν: φ and S increase.

For rolling curves to the imaginary spherical ellipse:

By increasing ϑ {For the tan-branches: φ and S decrease. For the cot-branches: φ and S increase.

For rolling curves to the imaginary spherical hyperbola:

By increasing ϑ {For curves with real values of ν and the larger branch of curves with imaginary values of ν: φ decreases and S increases. For the smaller branch of curves with imaginary values of ν: φ increases and S decreases.

For rolling curves to the imaginary spherical conic sections when $e \to 0$:

By increasing ϑ {For the tan-branches: φ and S decrease. For the cot-branches: φ and S increase.

The above equations and rules for evaluating the arc length of spherical conic section rolling curves are tabulated in Appendix 7.

J. Résumé of the deduction and interrelationship of the different spherical conic section rolling curves

In order to provide a clearer survey of the deduction and interrelationship of all the spherical conic section rolling curves, a "genealogical tree" of the deduction and interrelationship of the rolling curves is given in Appendix 1, in which also the spherical circle is included. This "tree" comprises a total of 19 spherical rolling curves of which 5 are crown-wheel curves. Instead of beginning from the spherical conic sections and deducing their rolling curves, it is also possible to start from the 5 crown-wheel curves and deduce the rolling curves to them.

CHAPTER VI

CONSTRUCTION OF SPHERICAL ROLLING CURVE PAIRS, i. e. PITCH CURVES FOR NON-CIRCULAR BEVEL GEARS, BY MEANS OF SPHERICAL CONIC SECTION ROLLING CURVES

A. The replacement of a given spherical rolling curve by a series of arcs of spherical conic section rolling curves

Any given spherical curve can always be replaced, approximately, by a series of arcs of spherical conic section rolling curves. For this purpose the given curve is divided into a more or less arbitrary number of monotonic arcs (i.e. r should be either always increasing or always decreasing with the polar angle) and each of these arcs is replaced by an arc of a spherical conic section rolling curve.

All spherical conic section rolling curves may be written as

$$\tan r = \frac{k_1}{k_2 - f(\vartheta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (234)$$

for which $f(\vartheta)$ has different significance for curves of different types, as may be seen from previously deduced equations, namely

eqs. (112), (123), (175) and (185)	for rolling curves with real ν-values to the real spherical conic section and the imaginary spherical hyperbola, where we have $f(\vartheta) = \cos \nu \vartheta$
eqs. (118), (128), (180) and (190)	for curves of the same type but with imaginary ν-values, where we have $f(\vartheta) = \cosh \lvert\nu\rvert \vartheta$
eqs. (147), (151), (156) and (163)	for rolling curves to the imaginary spherical ellipse, where we have $f(\vartheta) = \sinh \lvert\nu\rvert \vartheta$
eqs. (208) and (212)	for rolling curves to the limiting curve for the imaginary spherical conic sections when $e \to 0$ where we have $f(\vartheta) = \pm e^{\mp \lvert\nu\rvert \vartheta}$
eqs. (133) and (195)	for rolling curves to the imaginary spherical conic sections if $\nu = 0$, where we have $f(\vartheta) = -\vartheta^2$
eqs. (215)	for the same type of curve with $\lvert\nu\rvert = 0$, where we have $f(\vartheta) = \mp \vartheta$
eqs. (135)	for the spherical circle where we have $f(\vartheta) = 0$.

These equations include all types of spherical conic section rolling curve. We have a total of 7 different cases for the meaning of $f(\vartheta)$ corresponding to the 7 cases for plane conic section rolling curves listed in NCCG (Ref. 12) on page 88. (Compare also Appendix 3 in NCCG).

Furthermore, if r for the plane conic section rolling curves is replaced by $\tan r$, we obtain the equation for the corresponding spherical conic section rolling curves.

The given arc is assumed to be expressed in the form

$$\tan R = R(\Theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (235)$$

where we use the symbols (see the figure in Appendix 2)

R = the radius vector which is always taken positive and $0 < R < 180°$

Θ = polar angle

Further, we introduce

$\Delta\Theta = \Theta_e - \Theta_s$, where s and e are suffixes which refer to the beginning and end of the arc.

Because the angle Φ from the radius vector to the normal for plane curves is given by

$$\tan \Phi = -\frac{1}{R} \cdot \frac{dR}{d\Theta}$$

and for spherical curves by (compare eq. (54))

$$\tan \Phi = -\frac{1}{\sin R} \cdot \frac{dR}{d\Theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (236)$$

it can be understood that, if for spherical curves we introduce an auxiliary angle $\bar{\Phi}$ defined by

$$\tan \bar{\Phi} = -\frac{1}{\tan R} \cdot \frac{d(\tan R)}{d\Theta} = \frac{\tan \Phi}{\cos R} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (237)$$

then the formulae for the plane curves according to Appendix 3 in NCCG (Ref. 12) can be used for the deduction of the substitute arc also for spherical curves if R is replaced by $\tan R$, and Φ is replaced by $\bar{\Phi}$. The substitute arc and the given arc have the same absolute value of the radius vector as well as the same value of $\tan \bar{\Phi}$ and the same value of Θ at the beginning and end of the arc. Since the sign for k_1, and in some cases also for k_2, can be chosen arbitrarily, we must, however, observe that the signs are selected in such a way that they correspond to the equations previously derived for spherical conic section rolling curves.

We can now directly prepare a schedule for the deduction of the substitute spherical conic section rolling curve in the form

$$\tan r = \frac{k_1}{k_2 - f(\vartheta)}$$

by making a comparison with the corresponding schedule Appendix 3 in NCCG (Ref. 12). This schedule for spherical curves is given in Appendix 2.

If we put $\tan R = z$ and plot the curve $z = R(\Theta)$ in plane polar coordinates, we see from eq. (237) that $\bar{\Phi}$ is the angle from the radius vector z to the normal. This procedure therefore gives a method for determining $\bar{\Phi}$. If the given curve is projected radially from the centre of the sphere on to a tangent plane at the origin, we obtain the curve $z = R(\Theta)$. Therefore $\bar{\Phi}$ is the angle between the radius vector and the normal for this curve.

The angles Θ for the given curve and ϑ for the substitute curve are not the same. The relation between them can be obtained from

$$\vartheta - \vartheta_s = \Theta - \Theta_s \text{ or } \Theta = \vartheta - \vartheta_s + \Theta_s \quad \ldots\ldots\ldots\ldots\ldots\ldots (238)$$

After the substitute curve has been determined in the manner described above, we must determine the constants α_r, a_r or $\bar{a}_r$, α_1, a_1 and $\varkappa_r$. Formulae for this are given for all the different types of curve in the above-mentioned eqs. (112), (123), (175), (185), (118), (128), (180), (190), (147), (151), (156), (163), (208), (212), (133), (195), (215) and (135). These formulae have been included in Appendix 2, which also includes special columns giving the numbers of the equations in polar coordinates both in parametric and non-parametric form and the type of rolling curve obtained. Appendix 2 will therefore serve as a complete calculating schedule for determining the substitute arc for a given spherical arc. From the appendix it is evident that it is always possible to find a substitute arc to any given spherical arc. Since an arbitrary spherical curve can be divided into a series of monotonic arcs, an arbitrary spherical curve can always be replaced by a series of arcs of spherical conic section rolling curves, in which case both R and Φ are continuous on the limits between the different arcs. After having determined the equation of the substitute arcs by using Appendix 2, we still do not know if it is the main curve or the antipode curve which corresponds to the given curve. This can, however, be determined easily by using the rules given in Appendix 3. For rolling curves to the imaginary spherical ellipse and to the imaginary conic section when $e \to 0$, this is most easily determined by checking the sign for $\tan \Phi = \tan \bar{\Phi} \cos R$. It must be noted that, although R and r must have the same absolute values, they can have different signs. For rolling curves to the real spherical ellipse and to the imaginary spherical hyperbola it is therefore necessary at first to determine the value of r at the beginning or end of the arc, by using the equation obtained from Appendix 2, and compare the result with the value of R. If R lies in the first quadrant, then r must lie in the first or fourth quadrant, and if R lies in the second quadrant, then r must lie in the second or third quadrant.

If now the given spherical curve has been replaced by a series of arcs of conic section rolling curves, we can form with this substitute curve a mating curve with an arbitrary angle between the axes of rotation, likewise consisting of arcs of spherical conic section rolling curves. This can be seen from the fact that, according to the previous deductions of the rolling curves, one can always form a mating curve with arbitrary angle between the axes of rotation to any spherical conic section rolling curve. This mating curve is itself a spherical conic section rolling curve.

When replacing an arc of a given spherical curve by an arc of a spherical conic section rolling curve, it is necessary to know the angles $\bar{\Phi}_s$ and $\bar{\Phi}_e$, i.e., the angle $\bar{\Phi}$ at the beginning and end of the given arc. If the arc is given in analytical form, then $\bar{\Phi}$ can be determined by using eqs. (236) and (237). If the arc is only given empirically (for example, by a table showing the relations between R and Θ), $\dfrac{dR}{d\Theta}$ can be determined graphically, after which $\tan \bar{\Phi}$ can be determined from eqs. (236) and (237). Alternatively, we can plot the curve $z\ (= \tan R) = f(\Theta)$ in polar coordinates, in which case $\bar{\Phi}$ is the angle from the radius

vector to the normal to this curve. Usually, however, u or $\frac{1}{u} = f(\Theta)$ is given either analytically or graphically. For this case a formula giving $\tan \bar{\Phi}$ as a function of u and $\frac{du}{d\Theta}$ or $\frac{1}{u}$ and $\frac{d\left(\frac{1}{u}\right)}{d\Theta}$ will be derived.

According to eq. (237) we have

$$\tan \bar{\Phi} = -\frac{1}{\tan R} \cdot \frac{d(\tan R)}{d\Theta}$$

but we also have from eq. (6)

$$\frac{1}{u} = -\frac{\sin R}{\sin (|C_{L-R}| - R)}$$

where u = the instantaneous gear ratio

From this we can solve for $\tan R$, obtaining

$$\tan R = \frac{\sin |C_{L-R}|}{\cos |C_{L-R}| - u}$$

which together with the above expression for $\tan \bar{\Phi}$ gives

$$\tan \bar{\Phi} = -\frac{1}{\cos |C_{L-R}| - u} \cdot \frac{du}{d\Theta} = \frac{1}{\left(\frac{1}{u}\right)^2 \cos |C_{L-R}| - \frac{1}{u}} \cdot \frac{d\left(\frac{1}{u}\right)}{d\Theta} \quad \ldots\ldots (239)$$

It should be noted that u is negative for external meshing.

We now determine, analytically or graphically, u and $\frac{du}{d\Theta}$ or $\frac{1}{u}$ and $\frac{d\left(\frac{1}{u}\right)}{d\Theta}$, after which $\tan \bar{\Phi}$ can be determined by using eq. (239).

The angle ϑ_R is the independent variable in the equations of the arcs of the R-curve and is measured from the different polar axes for the separate arcs. Instead of ϑ_R we can introduce the angle Θ_R, which is measured from a common polar axis. This axis can be arbitrarily located, for example so that $\Theta_R = 0$ when $\Theta_L = 0$. Between Θ and ϑ there obviously applies the relation (238).

B. Determination of a single-turn mating curve to a given spherical rolling curve consisting of arcs of spherical conic section rolling curves

When determining the mating curve to a given spherical conic section rolling curve, the values of the given curve are given the suffix L and those of the required curve the suffix R. If the angle C_{L-R} between the centres of rotation is given, we can follow Appendices 3 and 4 when determining the mating curve. Using the equations in Appendix 3 we determine at first $\varkappa_R$ and $a_{rR} = \varkappa_R a_1$, or $|\varkappa_R|$ and $a_r = |\varkappa_R| |a_1|$, or $\bar{a}_r = 90° - |\varkappa_R| |a_1|$,

as well as the relation between φ_L and φ_R and if the mating curve is a main or an antipode curve. One solution only is obtained if we observe that a_1 and a_ν or $\bar{a}_\nu$ must be $< 90°$. Then we use Appendix 4, which comprises a compilation of the different spherical conic section rolling curves, for determining α_ν and ν or $|\nu|$, as well as the equation number of the mating curve. If the given curve consists of arcs of spherical conic section rolling curves, it is possible in this way to determine all the arcs of spherical conic section rolling curves building up the mating curve.

If we have a given single-turn spherical rolling curve and determine its mating curve for a given angle between the centres of rotation, then usually the mating curve will not be a single-turn curve. For a given mean gear ratio and a given kind of meshing, this can only occur with one single definite angle between the centres of rotation. If the mean gear ratio is given, then the correct angle between the centres of rotation must first be determined, after which the mating curve can be definitively calculated in the manner explained above. Determination of the angle between the centres of rotation cannot generally be done except by inspection.

The angle of displacement of the mating wheel when the given wheel rotates one revolution is according to eq. (11)

$$\int_0^{360°} \frac{\sin r_L}{\sin (r_L - C_{L-R})}\, d\,\Theta_L = \frac{360°}{U_{L/R}}$$

where $U_{L/R}$ is the mean gear ratio with respect to the L-wheel and taken with appropriate sign (positive for internal meshing, negative for external meshing).

First of all, an approximate expression for the above integral can be obtained by means of Simpson's rule. For this purpose each period of the L-curve should preferably be divided into at least 8 equal parts. If each period of the gear wheel is divided into an even number of parts $= m$, and if the different boundary radii are denoted by the suffix 1, 2, 3 ... $(m + 1)$, and if the angle between the centres of rotation is denoted by the suffix 1 to indicate that it corresponds to the value in the first approximation, we have

$$\frac{\sin r_{L1}}{\sin [r_{L1} - (C_{L-R})_1]} + \frac{4 \sin r_{L2}}{\sin [r_{L2} - (C_{L-R})_1]} + \frac{2 \sin r_{L3}}{\sin [r_{L3} - (C_{L-R})_1]} +$$

$$+ \frac{4 \sin r_{L4}}{\sin [r_{L4} - (C_{L-R})_1]} + \ldots + \frac{\sin r_{L(m+1)}}{\sin [r_{L(m+1)} - (C_{L-R})_1]} = \frac{3m}{U_{L/R}} \quad \ldots . \quad (240)$$

where the values included are taken with appropriate signs.

From eq. (240) $(C_{L-R})_1$ is determined by inspection. By using the value thus obtained for $(C_{L-R})_1$, the mating curves for all the arcs of the L-curve are determined. After these have been determined, ϑ_{Rs} and ϑ_{Re} are calculated for all arcs with the aid of Appendices 3, 4, and 6. The angle of rotation of the mating curve when the given L-curve rotates one revolution is then $n_L \underset{\text{over 1 per.}}{\Sigma} (\vartheta_{Re} - \vartheta_{Rs})_1$, and thus the mean gear ratio

$$(U_{L/R})_m = \frac{360°}{n_L \underset{\text{over 1 per.}}{\Sigma} (\vartheta_{Re} - \vartheta_{Rs})_m} \quad \ldots \ldots \ldots \ldots \ldots \quad (241)$$

where m = suffix indicating the number of the approximation.

The value obtained for $(U_{L/R})_m$, however, does not generally equal the given value $U_{L/R}$.

In the case of circular bevel gears we have

$$U_{L/R} = \frac{\sin (\tau_L - C_{L-R})}{\sin \tau_L}$$

or

$$\frac{\tan C_{L-R}}{\tan \tau_L} = 1 - \frac{U_{L/R}}{\cos C_{L-R}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (242)$$

Therefore $\tan C_{L-R}$ is a linear function of $\frac{U_{L/R}}{\cos C_{L-R}}$. If it is assumed that this applies approximately also to non-circular bevel gears, we can obtain a more accurate value for C_{L-R} by means of interpolation or extrapolation between two adjacent pairs of values previously determined. If we observe that for $C_{L-R} = 0$ the mean ratio $U_{L/R} = +1$, we thus have from eq. (242) for the second approximation to which is given the suffix 2

$$\frac{\tan (C_{L-R})_2 - 0}{\tan (C_{L-R})_1 - 0} = \frac{\dfrac{U_{L/R}}{\cos (C_{L-R})_2} - 1}{\dfrac{(U_{L/R})_1}{\cos (C_{L-R})_1} - 1} \quad \text{or} \quad \frac{\sin (C_{L-R})_2}{\cos (C_{L-R})_2 - U_{L/R}} = \frac{\sin (C_{L-R})_1}{\cos (C_{L-R})_1 - (U_{L/R})_1}$$

or, after simplification,

$$\left.\begin{aligned} \sin [z - (C_{L-R})_2] &= U_{L/R} \sin z \\ \text{where} \qquad \tan z &= \frac{\sin (C_{L-R})_1}{\cos (C_{L-R})_1 - (U_{L/R})_1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (243)$$

from which $(C_{L-R})_2$ can be determined.

z is an auxiliary angle and in the case of circular bevel gears $z = \tau_L$.

In eqs. (243) all values are taken with appropriate signs.

With this new value for the angle $(C_{L-R})_2$ between the centres of rotation, all the constants in the arcs of the R-curve are re-calculated, and in addition the corresponding new values of $n_L \sum_{\text{over 1 per.}} (\vartheta_{Re} - \vartheta_{Rs})_2$ and $(U_{L/R})_2$ from eq. (241) are determined.

If this new value of $U_{L/R} = (U_{L/R})_2$ does not equal the given value of $U_{L/R}$ either, we must make another interpolation between the pairs of values last determined. If these are denoted by the suffixes $m-1$ and m, then for the next approximate value for $C_{L-R} = (C_{L-R})_{m+1}$ we obtain by interpolation from eq. (242)

$$\frac{\tan (C_{L-R})_{m+1} - \tan (C_{L-R})_{m-1}}{\tan (C_{L-R})_m - \tan (C_{L-R})_{m-1}} = \frac{\dfrac{U_{L/R}}{\cos (C_{L-R})_{m+1}} - \dfrac{(U_{L/R})_{m-1}}{\cos (C_{L-R})_{m-1}}}{\dfrac{(U_{L/R})_m}{\cos (C_{L-R})_m} - \dfrac{(U_{L/R})_{m-1}}{\cos (C_{L-R})_{m-1}}}$$

or, after simplification,

$$\left.\begin{aligned} \sin [z - (C_{L-R})_{m+1}] &= k_i \, U_{L/R} \sin z \\ \text{where} \qquad k_i &= \frac{\sin [(C_{L-R})_m - (C_{L-R})_{m-1}]}{(U_{L/R})_{m-1} \sin (C_{L-R})_m - (U_{L/R})_m \sin (C_{L-R})_{m-1}} \\ \tan z &= \frac{(U_{L/R})_{m-1} \sin (C_{L-R})_m - (U_{L/R})_m \sin (C_{L-R})_{m-1}}{(U_{L/R})_{m-1} \cos (C_{L-R})_m - (U_{L/R})_m \cos (C_{L-R})_{m-1}} \end{aligned}\right\} \quad \ldots\ldots \quad (244)$$

from which $(C_{L-R})_{m+1}$ can be determined. Using this last value of $(C_{L-R})_{m+1}$, the corresponding values of $n_L \underset{\text{over 1 per.}}{\Sigma} (\vartheta_{Re} - \vartheta_{Rs})_{m+1}$ and $(U_{L/R})_{m+1}$ are determined. This procedure is repeated until we obtain a value of $n_L \underset{\text{over 1 per.}}{\Sigma} (\vartheta_{Re} - \vartheta_{Rs})$ which is practically equal to $\frac{360°}{U_{L/R}}$. Generally this occurs in the third and often in the second approximation.

Finally, all arcs of the definite R-curve are determined.

For plane rolling curves we can alter the distance between the centres of rotation by enlarging or reducing the rolling curves in a constant proportion. Such a re-scaling of r_r is, however, not possible for spherical rolling curves because then the ratio curve will be changed. If one of the curves and the mean ratio are given, this corresponds to a single angle between the centres of rotation. However, this angle can be changed if $\sin r$ for both curves is altered in the same proportion. This can be seen from the fact that from eq. (6) we have

$$\frac{1}{u_{L/R}} = \frac{\sin r_L}{\sin r_R} = \frac{\sin r_L}{\sin (r_L - C_{L-R})}$$

where r and C_{L-R} are taken with appropriate signs. The ratio is, therefore, not changed as the result of such a re-scaling of the curves and the mating curve will still be single-turn.

The above expression can be transformed into

$$u_{L/R} = \cos C_{L-R} - \frac{\sin C_{L-R}}{\tan r_L} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (245)$$

Since $u_{L/R}$ shall be kept constant, this means that

$$\cos C_{L-R} - \frac{\sin C_{L-R}}{\tan r_L} = \text{constant}$$

If the rolling curves are re-scaled in this way and if the new angle between the centres of rotation is denoted by $\bar{C}_{L-R}$, and the new radius vector by $\bar{r}$, we have

$$\cos C_{L-R} - \frac{\sin C_{L-R}}{\tan r_L} = \cos \bar{C}_{L-R} - \frac{\sin \bar{C}_{L-R}}{\tan \bar{r}_L} \quad \ldots\ldots\ldots\ldots \quad (246)$$

It will now be proved that the new curves will also become spherical conic section rolling curves.

Since all spherical conic section rolling curves can be expressed by eq. (234), the given L-curve can always be written as

$$\tan r_L = \frac{k_1}{k_2 - f(\vartheta_L)} \text{ or } \frac{1}{\tan r_L} = \frac{1}{k_1}[k_2 - f(\vartheta_L)]$$

If $\tan r_L$ from this expression is substituted in (246), we obtain

$$\cos C_{L-R} - \frac{\sin C_{L-R}}{k_1}[k_2 - f(\vartheta_L)] = \cos \bar{C}_{L-R} - \frac{\sin \bar{C}_{L-R}}{\tan \bar{r}_L}$$

or

$$\frac{1}{\tan \bar{r}_L} = \frac{\sin C_{L-R}}{k_1 \sin \bar{C}_{L-R}} \left[\frac{k_1 (\cos \bar{C}_{L-R} - \cos C_{L-R})}{\sin C_{L-R}} + k_2 - f(\vartheta_L) \right]$$

or

$$\left.\begin{aligned} \tan \bar{r}_L &= \frac{\bar{k}_1}{\bar{k}_2 - f(\vartheta_L)} \\ \text{where} \\ \bar{k}_1 &= \frac{k_1 \sin \bar{C}_{L-R}}{\sin C_{L-R}} \\ \bar{k}_2 &= k_1 \frac{\cos \bar{C}_{L-R} - \cos C_{L-R}}{\sin C_{L-R}} + k_2 = \frac{\bar{k}_1}{\tan \bar{C}_{L-R}} - \frac{k_1}{\tan C_{L-R}} + k_2 \end{aligned}\right\} \quad \ldots \quad (247)$$

For the types of curve where $f(\vartheta) = \pm e^{\mp |r| \vartheta}$ or $f(\vartheta) = \mp \vartheta$ we must, however, rotate the coordinate axes in such a way that we obtain $k_2 = 1$ or 0 to enable the equations to correspond with those previously given. The new L-curve thus always becomes a spherical conic section rolling curve, and this must also be the case for the R-curve, as the mating curve to any spherical conic section rolling curve is itself a spherical conic section rolling curve.

C. Derivation of a single-turn spherical rolling curve pair when the ratio curve is given

When designing a non-circular bevel gear it is generally not the shape of one wheel that is given, as assumed in the preceding chapter, but instead the angle between the centres of rotation and the gear ratio curve. The determination of the corresponding spherical conic section rolling curves will now be shown.

First of all the radius vector for the driving wheel, the L-wheel, as a function of $u_{L/R}$ is determined from the following equation, obtained from eqs. (9)

$$\tan r_L = \frac{\sin C_{L-R}}{\cos C_{L-R} - u_{L/R}} \text{ or } \cot r_L = \cot C_{L-R} - \frac{u_{L/R}}{\sin C_{L-R}} \quad \ldots\ldots \quad (248)$$

or, if we limit ourselves to external gears,

$$\tan R_L = \frac{\sin |C_{L-R}|}{\cos |C_{L-R}| + |u_{L/R}|} \text{ or } \cot R_L = \cot |C_{L-R}| + \frac{|u_{L/R}|}{\sin |C_{L-R}|} \quad \ldots\ldots \quad (249)$$

where R_L is always taken positive.

Next the L-curve is divided into a more or less arbitrary number of monotonic arcs, and the values of R_{Ls} and R_{Le} at the beginning and end of the different arcs are determined. Further, $\tan \bar{\Phi}_s$ and $\tan \bar{\Phi}_e$ for the different arcs must be determined. If the gear ratio curve or the reciprocal gear ratio curve is given analytically, $\tan \bar{\Phi}_s$ and $\tan \bar{\Phi}_e$ can be determined from eq. (239). If the reciprocal ratio curve is only known graphically, it is

possible to determine $\frac{1}{u_{L/R}}$ and $\frac{d\left(\frac{1}{u_{L/R}}\right)}{d\,\Theta_L}$ graphically, and insert them in the equation. The substitute curve for the L-curve consisting of arcs of spherical conic section rolling curves can now be determined by means of Appendix 2. The expressions thus obtained should, however, only be regarded as preliminary.

Next we determine all arcs of the mating curve with the aid of Appendices 3 and 4 in the same way as explained in the preceding section.

If we are concerned with a rotating gear, the R-wheel must rotate through an angle $\pm\, 360°/n_R$ when the L-wheel rotates through an angle $360°/n_L$. In the same way as shown in the preceding section, we therefore determine by means of Appendix 6 the angle of displacement of the R-wheel when the L-wheel is rotating 1 period $= 360°/n_L$. If the result does not equal $\pm\, 360°/n_R$, the curve preliminarily obtained must be adjusted.

From eqs. (234) and (6) we can derive

$$-\frac{1}{u_{L/R}} = \frac{k_1/\sin C_{L-R}}{k_2 - \dfrac{k_1}{\tan C_{L-R}} - f(\vartheta_L)}$$

which means that if $\bar{k}_1$ is substituted for k_1 and $\bar{k}_2$ for k_2, where $\bar{k}_1$ and $\bar{k}_2$ are determined by

$$\left.\begin{aligned} \bar{k}_1 &= \frac{k_1}{c} \\ \bar{k}_2 - \frac{\bar{k}_1}{\tan C_{L-R}} &= k_2 - \frac{k_1}{\tan C_{L-R}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (250)$$

we obtain a new spherical conic section rolling curve of such a nature that, for a given value of C_{L-R}, all the values of $u_{L/R}$ are changed in proportion to c.

From what is mentioned above it may be concluded that, if all the values of $u_{L/R}$ in the original ratio curve are replaced by $\bar{u}_{L/R}$ obtained from

where

$$\left.\begin{aligned} \bar{u}_{L/R} &= c\, u_{L/R} \\ c &= \frac{n_R}{360°} \sum_{\text{over 1 per.}} |\Theta_{Re} - \Theta_{Rs}| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (251)$$

and then the L-wheel is definitively re-calculated by using these new values, the mating wheel will also be single-turn.

When re-calculating the curves it should be observed that

$$\left.\begin{aligned} &k_3 \text{ remains unchanged} \\ &k_4 \quad \text{''} \quad \text{''} \\ &\nu \text{ or } |\nu| \text{ remains unchanged} \\ &\frac{\tan \bar{\Phi}_s}{\tan R_s} \text{ and } \frac{\tan \bar{\Phi}_e}{\tan R_e} \text{ are changed in the proportion } c \\ &\frac{\tan R_s \tan R_e}{\tan R_s - \tan R_e} \text{ is changed in the proportion } \frac{1}{c} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (252)$$

If the relations (252) are used, the re-calculation will be rather easy to perform. Eqs. (250) cannot, however, generally be used because the constants k_1 and k_2 in this equation may deviate from the constants in Appendix 2 in the derivation of which the polar axis was rotated in the case of rolling curves to the imaginary spherical conic section when $e \to 0$, and the sign for k_1 or the sign for k_2 and the value of ϑ_s were changed when k_2 passed 0.

After the equations of all the arcs forming the L-curve have thus been definitively derived, we must also determine from Appendix 3 which of the arcs are main curves and which are antipode curves. Finally, the definitive R-curve is deduced with the aid of Appendices 3 and 4.

Care should be taken to ensure that the given reciprocal ratio curve will at least approximately satisfy condition (11)

$$\int_0^{360^\circ} \frac{1}{u_{L/R}}\, d\,\Theta_L = \frac{630^\circ}{U_{L/R}}$$

However, this condition need not be exactly satisfied, since a deviation is automatically adjusted by means of the re-calculation described above.

D. Spherical twin rolling curves

In the following we shall confine ourselves to single-turn external twin gears, where $0 < |C_{L-R}| < 180^\circ$ and $|u| = -u$. The condition for twin wheels is then given by eqs. (20), where the polar axis must be located so that eq. (19) is satisfied.

Since the twin condition eqs. (19) and (20) for spherical rolling curves are the same as for plane curves according to eqs. (27) and (28) in NCCG (Ref. 12), the deduction of the definitive ratio curve will be identical with the corresponding deduction for cylindrical gears, which has been described on pages 119—122 in NCCG. If, for example, we desire the reciprocal ratio curve shown by a chain-dotted line in Fig. 11, which is the same as Fig. 31, page 120 in NCCG, then the unbroken line in the previously mentioned figure is practically the best result which can be obtained. This curve is determined in the following way. At first we check that the curve satisfies the condition (19). If this is not the case, the curve is first adjusted by means of a translation of the coordinate axes, or by increasing or decreasing the ratio curve in a constant proportion so that the condition will be approximately satisfied. The curve shown in Fig. 11 satisfies approximately the condition (19).

By using eqs. (20), we now deduce $\frac{1}{u_T\,(\Theta_R)}$, i.e. $\frac{1}{u_T}$ as a function of Θ_R. This function need not be determined accurately and therefore the integral may be determined relatively approximately. The curve $\frac{1}{u_T\,(\Theta_R)}$ has been determined and is plotted in the figure with a broken line.

If now for the left-hand part of the half period, (the wheel half), we adhere to the given chain-dotted reciprocal ratio curve, we then obtain for the right-hand part of the half

period the reciprocal ratio curve shown by a broken line. If, on the other hand, we adhere to the given reciprocal ratio curve for the right-hand part of the half period, we then obtain for the left-hand part of the half period the ratio curve shown by the broken line. The definite ratio curve must therefore lie somewhere between the two curves. The unbroken curve in the figure shows such a compromise curve. The least deviation from the required curve should be obtained if for the left-hand part we take the compromise curve as the geometric mean value to the chain-dotted curve and the broken curve. The ratio curve for

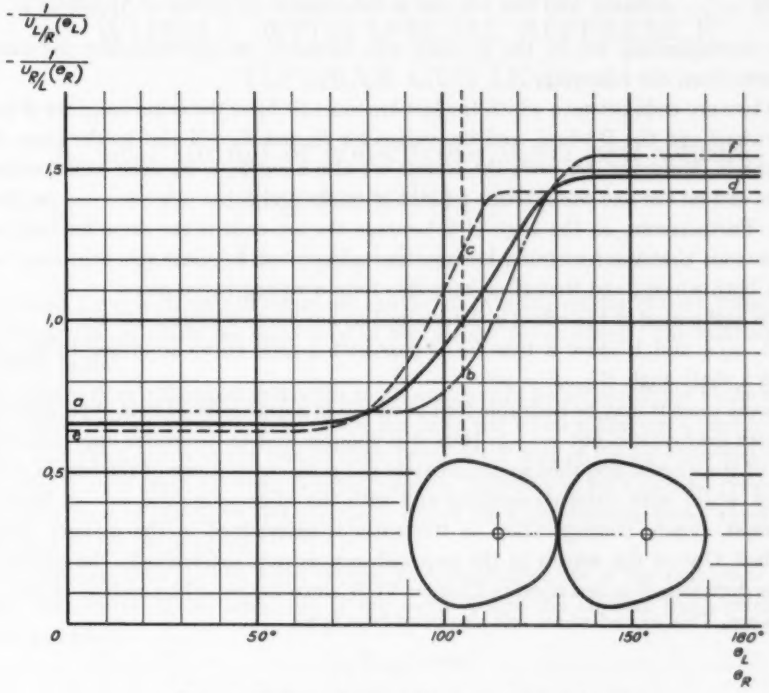


Fig. 11. Reciprocal gear-ratio curve for a twin gear.

the right-hand part is then obtained from eqs. (20). The limit between the left-hand and the right-hand part of the half wheel corresponds to the point where u_T passes through the value -1.

Determination of the spherical twin rolling curves which correspond to the compromise ratio curve obtained according to the foregoing, is done in a way similar to that previously used for determining the wheels corresponding to a given ratio curve. First of all we choose a definite angle C_T between the centres of rotation where $0 < C_T < 180°$. Then the half period is divided into a more or less arbitrary *odd* number of arcs. The arcs are numbered so that arcs with the same number on the L- and the R-curve can mesh. The number of arcs for a half period equals m. The extent of the last $\frac{m+1}{2}$ arcs of the L-wheel and of the

corresponding arcs of the R-wheel are so far unknown. By using eqs. (249) and (239), we first determine the values of R_s, R_e, $\tan \bar{\Phi}_s$ and $\tan \bar{\Phi}_e$, after which the equations for the different arcs can be deduced with the aid of Appendix 2.

We first determine the first arc of the L-wheel. This arc is identical with the arc number m (i.e. the last arc) on the R-wheel. We then determine arc number 1 on the R-wheel that meshes with arc number 1 on the L-wheel. This former arc is identical to arc number m on the L-wheel. The procedure is the same with the other arcs until only the arc with the number $\frac{m+1}{2}$ remains, and this arc also is determined by means of Appendix 2.

The corresponding arc on the R-wheel will, however, be automatically the same arc, as is seen from the following.

The two arc ends between which the last arc must fit have the same value for Φ for both the L-wheel and the R-wheel, and the values for R_s and R_e will also be the same for the two wheels. Moreover, for both the wheels we also have $\Phi_s = \Phi_e$ since, when rolling, the one arc end on the one wheel forms a point of contact with the other arc end on the other wheel. Furthermore, as the angle $\Delta \Theta$ between the arc ends is the same for both wheels, it is obvious that the remaining arc can be replaced by the same spherical conic section arc on both wheels, and that these arcs also form a rolling curve pair.

Each half period of the wheel is determined as has been described above.

Figs. 29 a and b show a twin bevel gear with a ratio curve according to Fig. 11 and having a shaft angle $|C_{L-R}| = 90°$.

In some cases it may be desirable for the wheels in a gear to be of twin type, even though the mean gear ratio is not $= -1$. Here it is possible first to determine approximately the shape of the wheels, and then approximately the ratio curve if one of the curves rolls with another wheel with external meshing and with the mean gear ratio $= -1$. Next a twin gear most closely corresponding to this ratio is determined in the manner previously described. One of the wheels in the required gear is now determined. The second wheel may be determined as described in Chapter VI:B, since the periodic numbers of both wheels are known. This second wheel will also be of twin type.

CHAPTER VII

MANUFACTURE OF NON-CIRCULAR BEVEL GEAR WHEELS, WITH SPECIAL REFERENCE TO GEAR CUTTING

A. Gear cutting of non-circular bevel gear wheels, using disc milling cutters

a) General description of method of manufacture

The teeth for non-circular bevel wheels can be cut by using milling cutters, (thinner than normal cutters) in much the same way as for circular bevel wheels. The method will be described below.

The centre lines if the teeth and of the tooth spaces must lie perpendicular to the spherical pitch curve, as can be understood if one imagines the wheel rolling on a circular crown-wheel. The pitch curve of the non-circular bevel wheel and of the crown-wheel are then tangential to each other, i.e. their normals coincide. Similarly as for a circular crown-wheel, the centre lines of the teeth and the tooth spaces for a non-circular bevel wheel lie perpendicular to the pitch curve.

As is the case with circular bevel wheels, the calculations are in the first place referred to the pitch curve at the large end of the teeth, where the circular tooth pitch $= m\pi$. We thus obtain

$$S_{pc}\, R_{sph} = m\pi Z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (253)$$

where

S_{pc} = the whole perimeter of the pitch curve in radian measure
m = the module
R_{sph} = the radius of the sphere on which the pitch curve lies = the length of the generatrix of the pitch cone
Z = the number of teeth

If the circumference is not the one given by eq. (253), the radius of the sphere, i.e. the length of the generatrix of the rolling cone, must be enlarged or reduced in order that eq. (253) may be satisfied, or alternatively, the circumference be slightly, for example 1/10 per cent, on the small side in order that the teeth may not be larger than the nominal thickness.

If the wheels are of twin type, it is desirable to have them identical. This will be the case if the teeth are so located that the end point of the arithmetic mean spherical radius vector coincides with the intersection of a tooth profile and the pitch line.

As the radius of curvature for a non-circular bevel gear varies along the circumference, the same cutter cannot be used around the whole circumference when cutting tooth by tooth with a disc cutter, but a cutter must be selected suitable for the number of teeth corresponding to the radius of curvature of the pitch curve in the middle of the tooth space concerned.

The involute tooth form for circular gear wheels corresponds for bevel gear wheels to a spherical involute profile. For bevel gears, however, a different kind of tooth form,

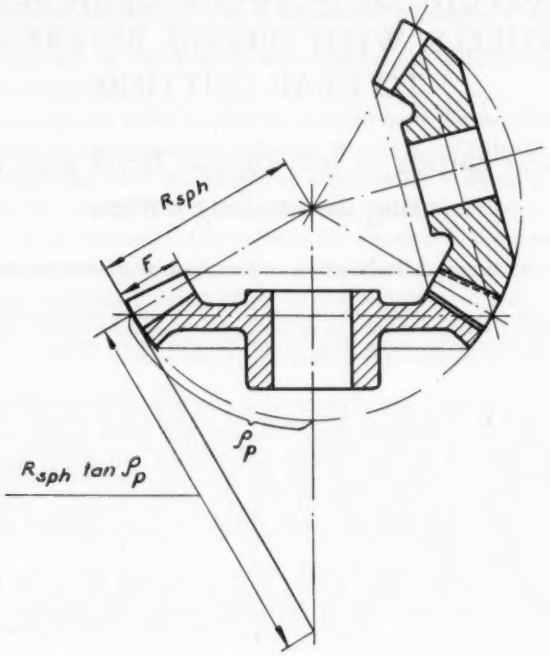


Fig. 12. When milling bevel wheels, a milling cutter having the radius of its pitch circle $= R_{sph} \tan \varrho_p$ should be used.

known as octoid form, is often used. This type of tooth form is obtained if the teeth are generated by a circular crown-wheel with straight-sided teeth. From Fig. 12 it should be clear that the profile for this kind of tooth in a section perpendicular to the generatrix of the pitch cone will approximate very closely the involute tooth form, having a radius of the pitch circle $= R_{sph} \tan \varrho_p$. As the radius of the pitch curve of a given cutter $= \frac{mZ_c}{2}$, the number of teeth Z_c for the cutter to be used is determined by

$$Z_c = \frac{2\,R_{sph}}{m} \tan \varrho_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (254)$$

where

Z_c = the number of teeth for which the cutter is designed when milling cylindrical gears

ϱ_p = the spherical radius of curvature of the pitch curve

Later in the text approximate expressions for tan ϱ_p and Z_c will be derived.

After preparation of the wheel blank and machining of its contour, the cutting of the teeth can proceed in the following manner.

First erect on the work table of the milling machine a rotary milling table. Then place a dividing head concentrically on the rotary table. Finally, set up the wheel blank on the arbor of the dividing head as is shown in Figs. 13 and 14. The index plate of the rotary milling table can be seen at the lower left-hand side of Fig. 14, and above it the index plate of the dividing head. The original index plates are both replaced by special plates graduated in degrees. The position of the blank in relation to the work table can now be

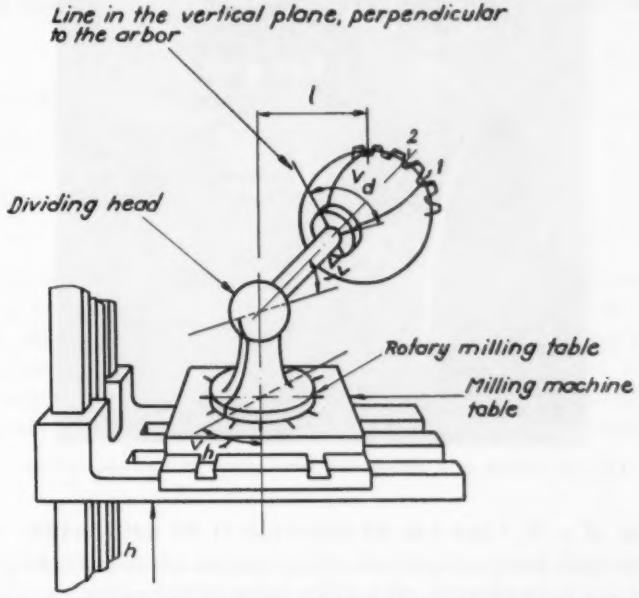


Fig. 13. Sketch explaining the settings v_d and v_v for the dividing head, v_h for the rotary milling table, h for the height setting and l for the cross setting of the milling machine table.

varied by means of three different settings, namely, the two index plates and the tilting of the arbor of the dividing head. An arbitrarily chosen generatrix on the pitch cone of the wheel as also the normal to the pitch cone can therefore be set in an arbitrary direction.

Further, we can adjust the elevation and cross-setting of the work table in such a way that the cutter cuts to an arbitrary depth and at an arbitrary distance from the centre of the dividing head. It is now obvious that by suitably selecting the five settings, one can cut an arbitrary tooth profile with the same degree of accuracy as when cutting circular bevel wheels, i.e. in such a way that the points of the tooth profile, which lie on the pitch cone, will be correct both at the larger end and the smaller end of the tooth. To attain this, however, the value of φ in the middle of the tooth space to be cut must be calculated first of all, and also the corresponding values of the spherical radius vector r, the polar angle ϑ, and the angle Φ from the radius vector to the normal to the pitch curve.

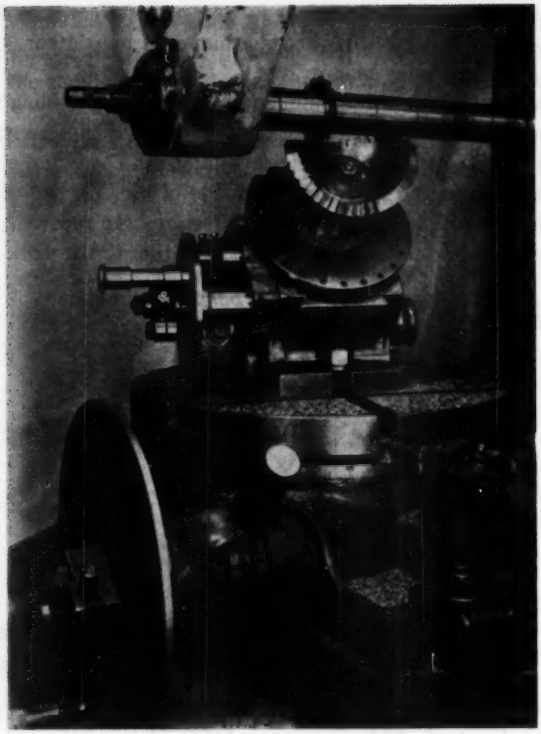


Fig. 14. Cutting of a non-circular bevel wheel with milling cutters.

b) Calculation of φ, Φ, r and ϑ at the midpoints of the teeth and the tooth spaces

The equation of the pitch line or an arc of the pitch line is assumed to be given in parametric form. Using the formulae in Appendix 7, we must first evaluate a table giving the arc length as a function of φ for the different arcs in a way similar to that shown in Chapter IV:A d, pp. 28 and 29, for a spherical ellipse. The table need only be calculated for one quadrant. For φ-values in other quadrants the corresponding values of the arc length S can be obtained for reasons of symmetry. We then investigate values of S (in radian measure) which correspond to the midpoints of the different teeth and tooth spaces. The length of an arbitrary arc can be obtained from the calculated table if φ is known. Conversely, one can determine the φ-value corresponding to a given S-value.

For determining the φ-value for a given point on the pitch curve, for example, for the midpoint in the tooth space number n, we must determine first of all the S-value to this point, which is denoted by S_n. The real circular tooth pitch in radian measure is obtained from

$$\Delta S = \frac{S_{pc}}{Z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (255)$$

where ΔS = the circular tooth pitch in radian measure.

Note that ΔS is not always exactly equal to $\frac{m\pi}{R_{sph}}$. When determining the value of S_n, we must calculate first of all for the previous arc with smaller values of Θ the absolute value of the arc length in radian measure from the midpoint of the last tooth space to the end of that arc. This value is denoted by ΔS_r and is obtained from

$$\Delta S_r = |S_e \text{ for the preceding arc} - S_n \text{ for the last tooth space of the preceding arc}| \quad (256)$$

where S_e = the value of S at the end of the arc.

The length from the beginning of the arc in question to the midpoint of the tooth space number n will thus be $= n\,\Delta S - \Delta S_r$ and the value of S_n for the same tooth space is

$$S_n = S_s \mp (n\,\Delta S - \Delta S_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (257)$$

where S_s = the value of S at the beginning of the arc

the negative sign applying if S decreases with Θ and ϑ, i.e. if $S_e - S_s$ is negative
the positive sign applying if S increases with Θ and ϑ, i.e. if $S_e - S_s$ is positive

} Compare Appendix 7

S_n having thus been evaluated for the different teeth and tooth spaces, the corresponding values of φ can be obtained by means of interpolation in the table which has been determined, and which gives S as a function of φ. Here inverse higher degree interpolation must often be used. For this kind of interpolation see NCCG, p. 192 (Ref. 12). After the values of φ have been determined, one can also calculate the corresponding values of r and ϑ from the equation of the curve in parametric form and Φ from the formulae compiled in Appendix 5.

c) Calculation of the settings of the dividing head and the rotary milling table when milling the central cuts

First of all, the settings will be calculated assuming that a given generatrix on the pitch cone will come in the same direction as the longitudinal axis of the work table of the milling machine and that the normal plane to the pitch curve through the generatrix will be vertical.

Imagine at first that the wheel blank is not set up as shown in Figs. 13 and 14, but is moved along the arbor of the dividing head to the supposed negative side of that arbor so that the midpoint of the sphere on which the pitch curve lies coincides with the centre of the dividing head. Fig. 15 shows the sphere with the pitch curve lying on it as also a great circle tangential to the pitch curve at a given point P, and another great circle through the same point P and perpendicular to the pitch curve. Further, the spherical radius vector r to the point P on the pitch curve has been shown.

In the starting position the arbor of the dividing head is assumed to be parallel to the longitudinal axis of the work table with the spherical polar axis lying in the vertical plane. We can now rotate the sphere by means of the index handle of the dividing head through

an angle v_d, so chosen that the plane through the great circle tangential to the pitch curve intersects the work table perpendicularly to its longitudinal axis. This position is shown in the figure. After this the arbor of the dividing head is tilted through the angle v_v so that this circle comes in the horizontal plane. Finally, the rotary milling table is rotated through the angle v_h so that the radius of the sphere to the given point P on the pitch curve falls parallel to the longitudinal axis of the work table. We have now obtained the settings desired.

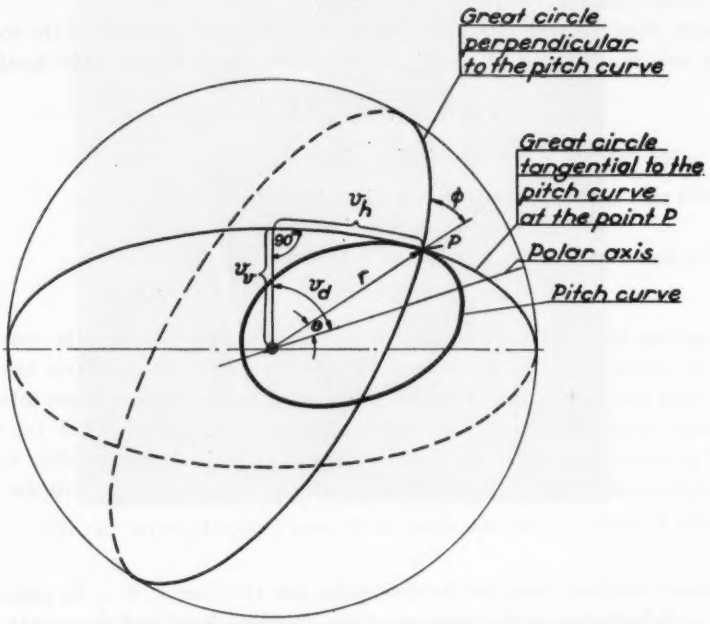


Fig. 15. The significance of the machine settings v_d, v_v and v_h when cutting the central cut of the teeth.

After the first rotation, we obtain the setting shown in Fig. 15. The angle between the radius vector r to the given point P and the great circle tangential to the pitch curve equals $90° - \Phi$. From the right-angled spherical triangle formed by the sides r, v_h and v_v we obtain, by using the spherical law of sines and two other well-known laws for right-angled spherical triangles,

$$\left.\begin{aligned} \sin v_v &= \sin r \cos \Phi \\ \tan v_h &= \tan r \sin \Phi \\ \tan (v_d - \Theta) &= \frac{\tan \Phi}{\cos r} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (258)$$

We shall now treat the case where the generatrix at the bottom of the theoretical tooth space will fall horizontally and have the same direction as the longitudinal axis of the work table and with the normal plane through the pitch curve falling in the vertical plane.

It is obvious that we obtain the same relations as given in the system of equations (258), although the great circle tangential to the pitch curve must now be replaced by the great circle at the bottom of the theoretical tooth space and perpendicular to the normal to the pitch curve. The radius vector r must therefore be replaced by the radius vector to the theoretical bottom of the tooth space. This radius vector is denoted by r_c. The angle Φ must be replaced by the angle from the radius vector r_c to the normal to the pitch curve. This latter angle is denoted by Φ_c. Finally, the angle Θ must be replaced by the angle to the radius vector to the bottom of the theoretical tooth space, which angle is denoted by Θ_c.

We now form expressions for these new magnitudes. From the spherical triangle in Fig. 16 we obtain from the spherical cosine law

$$\cos r_c = \cos B_{tk} \cos r + \sin B_{tk} \sin r \cos \Phi$$

where B_{tk} = the dedendum minus bottom clearance (in angular measure).

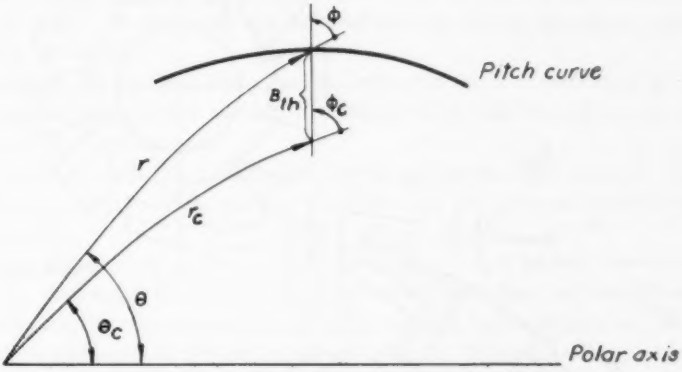


Fig. 16. The theoretical dedendum B_{tk} and the radii to the centre of the tooth space, r to the pitch curve and r_c to the theoretical bottom of the tooth space, form a spherical triangle from which r_c, Θ_c and Φ_c can be determined.

Furthermore, we obtain from the spherical law of sines

$$\sin \Phi_c = \frac{\sin r}{\sin r_c} \sin \Phi$$

and

$$\sin (\Theta - \Theta_c) = \frac{\sin B_{tk}}{\sin r} \sin \Phi_c$$

If now in the system of equations (258) we make the above substitutions, i.e. if we add to the magnitudes r, Φ and Θ the suffix c, and also add the same suffix to the magnitudes v_v, v_k and v_d to indicate that the values belong to the theoretical bottom of the tooth space, the system of equations (258) together with the last equations then give the following relations

$$
\left.
\begin{aligned}
&\cos r_c = \cos B_{th} \cos r + \sin B_{th} \sin r \cos \Phi \\
&\sin \Phi_c = \frac{\sin r}{\sin r_c} \sin \Phi \\
&\sin v_{vc} = \sin r_c \cos \Phi_c \\
&\tan v_{hc} = \tan r_c \sin \Phi_c \\
&\sin (\Theta - \Theta_c) = \frac{\sin B_{th}}{\sin r} \sin \Phi_c \\
&\tan (v_{dc} - \Theta_c) = \frac{1}{\cos r_c} \tan \Phi_c \\
&v_{dc} = (v_{dc} - \Theta_c) - (\Theta - \Theta_c) + \Theta
\end{aligned}
\right\} \quad \ldots\ldots\ldots\ldots\ldots (259)
$$

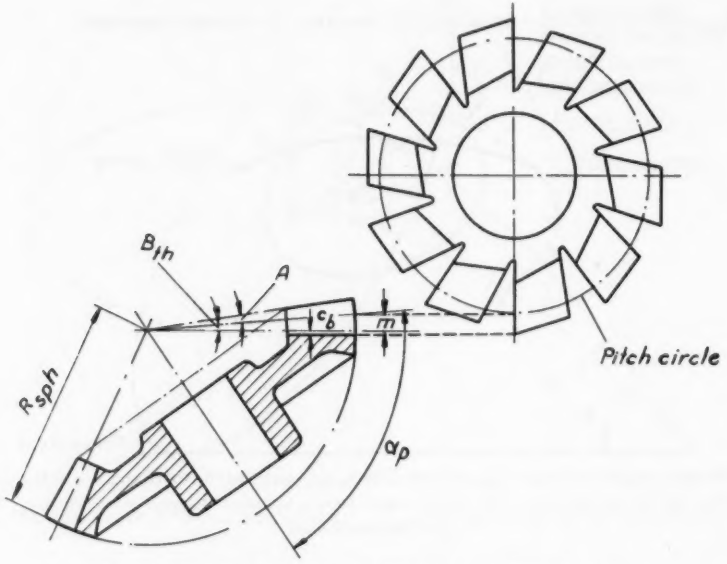


Fig. 17. Cutting bevel wheel with circular milling cutter.

For bevel gears, cut by milling cutters, we have the dedendum minus bottom clearance = m, if measured perpendicularly to the theoretical tooth bottom, see Fig. 17. If measured on the spherical surface on which the pitch curve lies, we thus have

$$\sin B_{th} = \frac{m}{R_{sph}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (260)$$

For the addendum A, measured in radians, we may choose the same value as for B_{th} or we may choose $A = \frac{m}{R_{sph}}$. The difference between these alternatives is, however, so small that it can be neglected.

We thus choose

$$\left.\begin{aligned} \sin A &= \frac{m}{R_{spk}} \\ \text{or alternatively} \\ A &= \frac{m}{R_{spk}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (261)$$

Although the wheel blank is not set up as postulated above, but moved along the arbor of the dividing head as shown in Figs. 13 and 14, the angles obtained from the above equations will not be affected because the generatrix in question going through the theoretical bottom of the tooth space will still be lying parallel to the longitudinal axis of the milling-machine table.

Eqs. (259), as are also all the following equations for the settings, are based on certain definite directions of movement for the dividing head, the rotary milling table and the elevation and cross-setting of the milling-machine table. In Fig. 14 the cutting of bevel-wheel teeth is shown. In eqs. (259) the different settings obtain the correct signs if the following rules are valid.

If the handle of the dividing head (for the setting of v_d) is turned towards increased setting (usually clockwise), then the arbor of the dividing head should rotate clockwise, as viewed from the dividing head.

The slope of the arbor of the dividing head to the milling-machine table (for the setting of v_v) is measured from the position where the arbor and the table are parallel and in the ascending direction.

If the handle of the rotary milling table for the setting of v_v is turned towards increased setting (usually clockwise), then the table should rotate clockwise, as viewed from above.

If the handle for the cross-setting (l) is turned towards increased setting, the milling-machine table should move to the left, as viewed from the dividing head.

If the handle for the elevation of the milling-machine table (for the height setting h) is turned towards increased setting, the table should be elevated.

If the directions of movement are opposite to those mentioned above, the signs for the corresponding quantities must be changed.

For $v_d = 0$ the radius vector plane is assumed to be lying in the vertical plane.

In the calculations we obtain v_d and v_h expressed in degrees. It is desirable, however, to have them expressed in revolutions+degrees on the index plates. If the ratio is 40/1 for the dividing head and 60/1 for the rotary table, which is generally the case, the desired value of v_d is most easily obtained by dividing v_d, expressed in degrees, by 9, in which case the integral quotient gives the number of revolutions of the dividing handle. If the remainder is then multiplied by 40, we obtain the surplus number of degrees. In a similar way the value of v_h is obtained by dividing v_h, expressed in degrees, by 6 and multiplying the remainder by 60.

d) Calculation of the elevation and cross-setting of the milling-machine table when milling the central cuts of the tooth spaces

If the wheel blank were erected as postulated above, i.e. so that its apex coincides with the centre of the dividing head, it would now be possible to mill the central cut of any tooth space with the settings of the dividing head and the rotary table according to eqs. (259). The milling cutter should then first be set directly above the centre of the arbor of the dividing head. The elevation of the milling-machine table should be chosen so that the pitch line of the milling cutter coincides with the pitch cone at the larger end of the tooth space, i.e. the highest point on the pitch line of the milling cutter should come a distance m above the centre of the dividing head, or the tip of the milling cutter a distance $d-2m$ below the centre of the dividing head, where

d = the whole depth of the tooth space, usually stamped on the cutter.

We have

$$d = 2m + c_b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (262)$$

where

c_b = the bottom clearance

When cutting, the wheel blank must, however, be set up as shown in Figs. 13 and 14 and not as assumed above. In order that the milling cutter may cut to the correct depth, the milling-machine table must be lowered a distance $= f \sin v_{vc}$ from the position stated above, and be given a cross-setting $l_c = f \cos v_{vc} \sin v_{hc}$ to the left, as viewed from the dividing head, where

f = the distance from the centre of the dividing head to the apex of the pitch cone

If the elevation and cross-setting of the milling-machine table are measured from the position where the tip of the cutter coincides with the centre of the dividing head, we obtain the following formulae for the settings of the milling-machine table

$$\left.\begin{aligned} \bar{h}_c &= d - 2m - f \sin v_{vc} \\ l_c &= f \cos v_{vc} \sin v_{hc} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (263)$$

In order to avoid negative values of the settings for the elevation of the work table, some other zero level than that passing through the centre of the dividing head may be chosen. The zero level is then conveniently chosen so that the height setting equals zero when milling the tooth space for which $v_{vc} = \max$. We can then proceed in the following way. Firstly, a circular setting disc having the diameter

$$D_c = 2\,[2\,m + f \sin (v_{vc})_{\max}] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (264)$$

is prepared and fitted on the arbor of the dividing head, the arbor being set parallel to the work table. The cross-setting of the milling-machine table is now adjusted so that the cutter comes directly above the centre of the arbor, the scale for the cross-feed then being set at zero and locked. The elevation of the work table is adjusted so that the cutter touches the setting disc. After this, the work table is elevated a distance equal to the cutting depth d stamped on the cutter (= the tooth height including bottom clearance), and the scale for the elevation is set at zero and locked.

The height setting h_c and cross-setting l_c when cutting an arbitrary tooth space will now become

$$\left.\begin{aligned} h_c &= \left[\frac{D_c}{2} - 2\,m\right] - f \sin v_{vc} \\ l_c &= f \cos v_{vc} \sin v_{hc} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (265)$$

where

D_c = the diameter of the setting disc, conveniently chosen according to eq. (264).

Because the arbor is, as a rule, fitted to the dividing head by means of a morse taper, it is difficult to determine the distance f before the arbor has been manufactured and fitted to the dividing head. Then, however, it is possible to measure the distance f_1 from

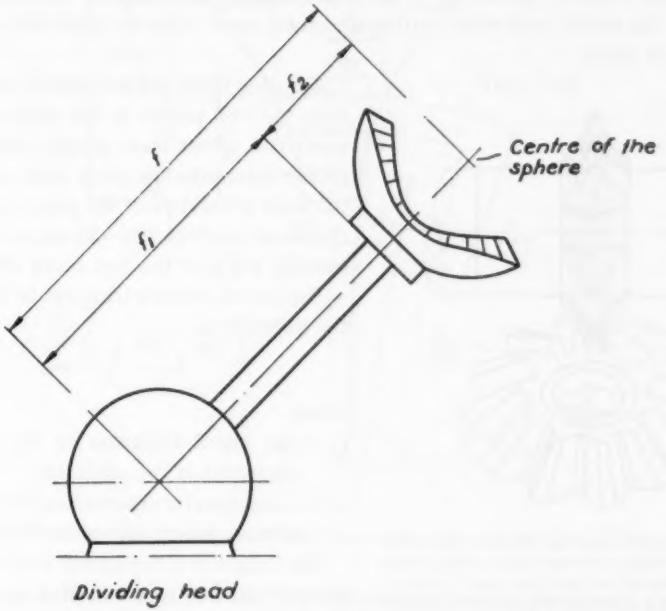


Fig. 18. The distance f from the centre of the dividing head to the centre of the sphere is obtained by measuring the distance f_1 from the centre of the dividing head to the locating face of the boss of the bevel wheel, and adding to this the distance f_2 from the locating face of the wheel boss to the apex of the pitch cone, which distance can be taken from the drawing of the wheel.

the centre of the dividing head to the locating face for the boss of the bevel wheel, and f may now be obtained by adding to this distance the distance f_2 from the outer side of the wheel boss to the apex of the pitch cone, as can be seen from Fig. 18.

The diameter D_c of the setting disc, as taken from eq. (264), may become absurd. It can, however, be chosen arbitrarily, and eqs. (265) will still be valid. But when it is necessary to reduce the diameter, it is convenient to decrease the value obtained from eq. (264) by a rounded-off value, for example, 200 mm.

After having determined the settings h_c and l_c of the work table for all tooth spaces, the central cuts can be milled if the number of teeth Z_c for which the cutter has to be chosen is known. How Z_c may be determined will be shown later on in the text.

e) Calculation of the settings when cutting the flanks of the teeth (the side cuts) for bevel gear wheels

1) Non-circular bevel gear wheels

The above derivations of the machine settings are valid for the central cuts. When milling the flanks of the teeth, the cross-setting of the milling-machine table must be adjusted so that the cutter is set out-of-centre a certain distance in the opposite direction to the flank being cut, in the same way as when cutting circular bevel wheels using milling cutters (see Fig. 19). Further, the setting of the rotary milling table must be changed by an angle Δv_{hc} from the setting used when cutting the centre cut in order to obtain the correct width of the tooth space.

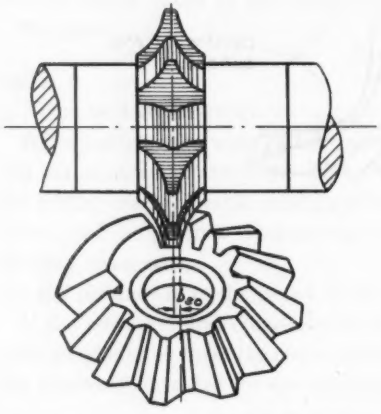


Fig. 19. When milling the flanks of the teeth, the cross setting of the milling machine table must be adjusted so that the cutter is set out of centre by a distance of b_{so} in the opposite direction to the flank being cut.

By using these new settings, it can be ensured that the two points at the large end and the small end of the tooth profile, where the tooth profile intersects the pitch cone, shall come on the same generatrix of the pitch cone. Although the whole tooth profile will not be correct, these settings will give the best result obtainable.

The out-of-centre setting can be obtained from the formula

$$b_{so} = \frac{g_p}{2} - mc_c \quad \ldots\ldots\ldots \quad (266)$$

where

g_p = the chord thickness of the cutter used, measured at the pitch line

c_c = a constant for determining the out-of-centre setting, which will be derived below

The values of c_c commonly used hitherto, and given by R. E. FLANDERS (Ref. 5) are, however, very erroneous and often up to 40 per cent too low if used for a pressure angle of 20°. A new derivation of c_c will therefore be made.

A standard cutter for the next lower number of teeth than that obtained from eq. (254) is chosen. The cutters must, however, be thinner than those used when cutting cylindrical gears to prevent too much material from being cut away at the small end of the teeth. The profiles, however, are the same as for ordinary milling cutters, but displaced so that the cutter is thinner.

By using a milling cutter selected according to eq. (254), we can obtain a tooth profile at the large end of the tooth which very closely approximates the correct profile. By choosing suitable machine settings, we can also ensure that the generatrix of the tooth at the pitch cone shall be fairly correct. The profile at the small end of the tooth will, however, deviate appreciably from the desired profile. If the deviation should become intolerable, the profile must be adjusted by means of filing. In fact, we cannot even obtain a fully correct generatrix at the pitch cone, as this will always become somewhat concave for

external teeth, but we can ensure that the two points at the large end and the small end of the tooth profile, where the tooth profile intersects the pitch cone, shall come on the same generatrix of the pitch cone. The attainment of this result has been the object also in FLANDERS' investigations.

The cutting of the flanks can now proceed in the following manner, as should be clear from Figs. 13 and 14.

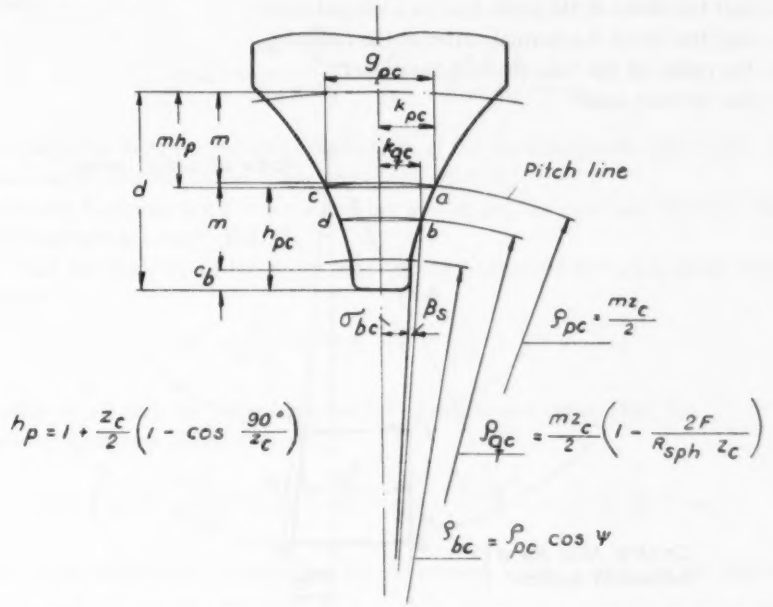


Fig. 20. Notations used for the different magnitudes of a normal disc milling cutter.

Suppose the settings are at first the same as when cutting the central cut, i.e. according to eqs. (259) and (265). The cross-setting of the milling-machine table is then moved a distance b_{so} so determined that the two points of intersection between the pitch cone and the large end and the small end of the required tooth profile come on the same generatrix of the pitch cone. Finally, the rotary milling table is rotated through a certain angle to obtain the correct width of the tooth space.

The radii of the cutter (see Fig. 20) which correspond to the above-mentioned two points are denoted by ϱ_{pc} and ϱ_{qc}. As ϱ_{pc} is the radius of the pitch circle of the cutter, we have

$$\varrho_{pc} = \frac{mZ_c}{2}$$

Further, we have with good approximation

$$\varrho_{qc} = \varrho_{pc} - \frac{m\,F}{R_{sph}} = \frac{m\,Z_c}{2}\left(1 - \frac{2\,F}{R_{sph}\,Z_c}\right)$$

where

F = the face width of the tooth.

Strictly speaking, this value of ϱ_{qc} is not exact, mainly because the two radii ϱ_{pc} and ϱ_{qc} form an angle, although very small, with one another, but, as will be shown in the following, the approximation is negligible, and further it is necessary, in order to prevent c_c from becoming a function of more than two independent variables.

The following additional notations are now introduced (see Fig. 20):

g_{pc} = the chord at the pitch line for a normal cutter
k_{pc} = half the chord at the pitch line for a normal cutter
k_{qc} = half the chord of a normal cutter at the radius ϱ_{qc}
ϱ_{bc} = the radius of the base circle of the cutter
ψ = the pressure angle

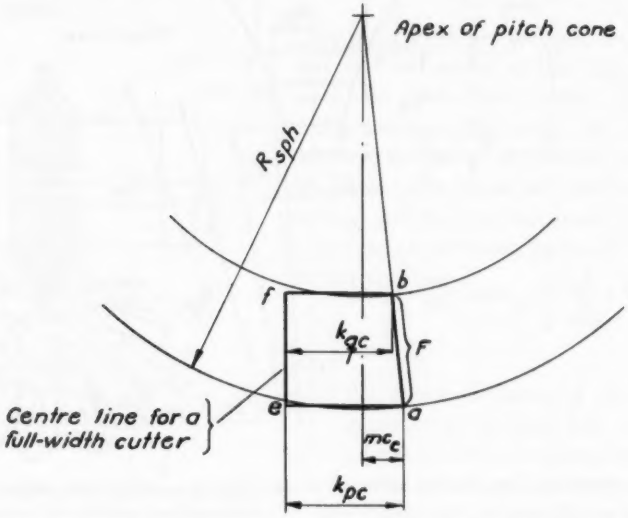


Fig. 21. Section perpendicular to the centre plane of the milling cutter and passing through the apex of the pitch cone and the intersection *a* between the pitch curve and the tooth profile.

We have

$$\varrho_{bc} = \varrho_{pc} \cos \psi = \frac{m Z_c}{2} \cos \psi$$

We now consider a tooth space being cut and lay a plane through the generatrix on the tooth profile and perpendicular to the centre plane of the cutter (see Fig. 21). This plane, as also the extension of the line *ab*, must pass through the apex of the pitch cone. As may be seen from Fig. 21, we therefore obtain

$$\frac{mc_c}{R_{sph}} = \frac{k_{pc} - k_{qc}}{F}$$

or

$$c_c = \frac{R_{sph}}{m F} (k_{pc} - k_{qc})$$

The values of k_{pc} and k_{qc} will now be derived. For k_{pc} we obtain the expression

$$k_{pc} = \frac{m Z_c}{2} \sin \frac{90^\circ}{Z_c}$$

For the determination of k_{qc}, the angle β_s (see Fig. 20) must at first be calculated. We obtain

$$\operatorname{ev} \beta_s = \frac{\varrho_{pc} - \dfrac{m F}{R_{sph}}}{\varrho_{pc} \cos \psi} = \left(1 - \frac{2 F}{R_{sph} Z_c}\right) \frac{1}{\cos \psi} \quad \ldots\ldots\ldots\ldots\ldots (267)$$

where

β_s = the angle between the radius to the bottom of the involute profile (the radius of the base curve) and the radius to the end of the chord k_{qc}

For involute functions (ev $\beta = \sec \alpha$ and inv $\alpha = \beta$) see, for example, PETERS: "Kreis- und Evolventenfunktionen" (Ref. 8).

Next, half the angle σ_{bc} of the tooth space at the base circle for a full-width cutter is determined.

$$\sigma_{bc} = \frac{\pi}{2 Z_c} - \tan \psi + \psi = \frac{\pi}{2 Z_c} - \operatorname{inv} \psi \quad \ldots\ldots\ldots\ldots\ldots (268)$$

The value of σ_{bc} may be taken from the tables mentioned above (Ref. 8).

The chord k_{qc} is now obtained from

$$k_{qc} = \left(\varrho_{pc} - \frac{m F}{R_{sph}}\right) \sin (\beta_s + \sigma_{bc}) = \frac{m F}{R_{sph}} \left(\frac{R_{sph} Z_c}{2 F} - 1\right) \sin (\beta_s + \sigma_{bc})$$

Finally, c_c is obtained by substituting the expressions for k_{pc} and k_{qc} in the expression for c_c

$$c_c = \frac{R_{sph} Z_c}{2 F} \sin \frac{90^\circ}{Z_c} - \left(\frac{R_{sph} Z_c}{2 F} - 1\right) \sin (\beta_s + \sigma_{bc}) \quad \ldots\ldots\ldots\ldots (269)$$

where β_s and σ_{bc} are taken from eqs. (267) and (268).

To determine the out-of-centre setting b_{so} from eq. (266) it is necessary to measure the thickness g_p at the pitch line of the cutter used. This may be carried out by using a gear-tooth vernier. As can be seen from Fig. 20, we shall then use a calliper setting of height

$$h_{pc} = d - m \left[1 + \frac{Z_c}{2} \left(1 - \cos \frac{90^\circ}{Z_c}\right)\right]$$

where

d = the whole depth of the tooth space $= 2\,m + c_b$

c_b = the bottom clearance

or

where

$$\left.\begin{aligned} h_{pc} &= d - m c_{pc} \\ c_{pc} &= 1 + \frac{Z_c}{2} \left(1 - \cos \frac{90^\circ}{Z_c}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (270)$$

The value of d is usually stamped on the cutter.

The values of c_c have been calculated for the pressure angle $\psi = 20°$ for different numbers of teeth Z_c and different values of $\frac{F}{R_{sph}} = \frac{\text{Length of tooth face}}{\text{Apex distance}}$ and have been tabulated and plotted as curves in Appendix 8. This also contains the values of c_{pc} for standard numbers of teeth Z_c as well as all other particulars necessary for determining the value of the out-of-centre setting b_{so}.

When cutting the side cuts, the setting of the rotary milling table must be increased (for the right-hand cut as viewed from the large end of the tooth) or decreased (for the left-hand cut) by an angle Δv_{hc} in order to obtain the correct width of the tooth space. An expression for this additional angle Δv_{hc} will now be derived. We place an osculating circle to the pitch curve at the large end of the teeth (see Fig. 22). This circle lies perpendicular to a line going through the apex and the centre of curvature for the pitch curve. Its radius is denoted by ϱ_{osc}. From Fig. 22 we have

$$\varrho_{osc} = R_{sph} \sin \varrho_p = \frac{m Z_{osc}}{2}$$

where

$Z_{osc} = Z_c \cos \varrho_p$ = the number of teeth for a circular bevel gear wheel where half the apex angle of the pitch cone is constant = ϱ_p. (The radius of the osculating circle to the pitch curve is constant = ϱ_{osc}.)

Half the tooth space covers on the osculating circle an angle $= \frac{m \pi}{4 \varrho_{osc}} = \frac{\pi}{2 Z_{osc}}$

Half the chord k_p of the tooth space at the pitch line is

$$k_p = \varrho_{osc} \sin \frac{\pi}{2 Z_{osc}} = \frac{m Z_{osc}}{2} \sin \frac{\pi}{2 Z_{osc}}$$

Half the angle v_x of the tooth space measured in the horizontal plane is therefore given by

$$\sin v_x = \frac{k_p}{R_{sph} \cos B_{th}} = \frac{m Z_{osc}}{2 R_{sph} \cos B_{th}} \sin \frac{\pi}{2 Z_{osc}}$$

In the denominator we should in fact have instead of B_{th} the angle between the horizontal plane and a plane through the apex and the intersection between the pitch curve and the tooth profiles. But as this angle is very small and nearly equal to B_{th}, the error, if this angle is replaced by B_{th}, is negligible.

The angle v_y, formed by the vertical plane through the generatrix of the pitch cone, generated by the milling cutter, and the vertical plane through the apex of the pitch cone and in the longitudinal direction of the cutter, is determined by

$$\sin v_y = \frac{m c_c}{R_{sph} \cos B_{th}}$$

Thus, to obtain the correct width of the tooth space, the rotary milling table should be rotated through an additional angle Δv_{hc}, determined by

$$\Delta v_{hc} = v_x - v_y$$

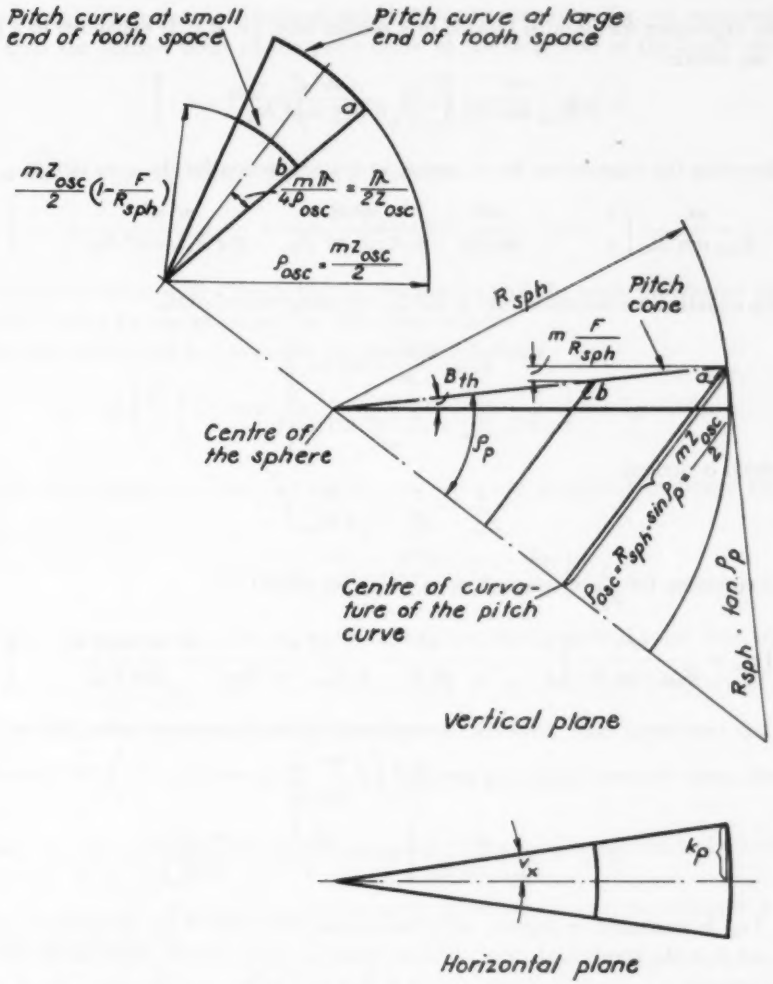


Fig. 22. Sketches for determining the desired depth below the highest point of the pitch curve for points *a* and *b*. The sketch applies to non-circular bevel gear wheels when a rotary milling table is used.

If $\sin\frac{\pi}{2Z_{osc}}$ is expanded in a power series in the above expression for $\sin v_x$, we obtain

$$\sin v_x = \frac{m}{R_{sph}\cos B_{th}}\left[\frac{\pi}{4} - \frac{\pi^3}{96\,Z_{osc}^2} + \cdots\right]$$

If now v_x is expanded, we obtain

$$v_x = \frac{m}{R_{sph}\cos B_{th}}\left[\frac{\pi}{4} - \frac{\pi^3}{96\,Z_{osc}^2} + \frac{m^2\pi^3}{384\,R_{sph}^2\cos^2 B_{th}} \cdots\right]$$

If the expression for $\sin v_y$ is treated in a similar way, i.e. if v_y is expanded in a power series, we obtain

$$v_y = \frac{m}{R_{sph} \cos B_{th}} \left[c_c + \frac{m^2 c_c^3}{6 R_{sph}^2 \cos^2 B_{th}} + \dots . \right]$$

By inserting the expressions for v_x and v_y in the expression for Δv_{hc}, we obtain

$$\Delta v_{hc} = \frac{m}{R_{sph} \cos B_{th}} \left[\frac{\pi}{4} - c_c - \frac{\pi^3}{96 Z_{osc}^2} - \frac{m^2 c_c^3}{6 R_{sph}^2 \cos^2 B_{th}} + \frac{m^2 \pi^3}{384 R_{sph}^2 \cos^2 B_{th}} + \dots \right] \quad (271)$$

In this equation we can substitute Z_c for Z_{osc} by using the relations

$$\left. \begin{aligned} Z_c &= \frac{2 R_{sph}}{m} \tan \varrho_p \\ Z_{osc} &= \frac{2 R_{sph}}{m} \sin \varrho_p \end{aligned} \right\}$$

from which is derived

$$\frac{1}{Z_{osc}^2} = \frac{1}{Z_c^2} + \left(\frac{m}{2 R_{sph}} \right)^2$$

If this expression for $\frac{1}{Z_{osc}^2}$ is inserted in eq. (271), we obtain

$$\Delta v_{hc} = \frac{m}{R_{sph} \cos B_{th}} \left[\frac{\pi}{4} - c_c - \frac{\pi^3}{96 Z_c^2} - \frac{c_c^3 m^2}{6 R_{sph}^2 \cos^2 B_{th}} + \frac{\pi^3 m^2 \tan^2 B_{th}}{384 R_{sph}^2} \dots \right]$$

The last two terms may, however, be neglected as they can never affect the width of the tooth space by more than 0.05 per cent $\left(\text{if} \frac{m}{R_{sph}} \leq 12 \text{ and } Z_c \geq 12 \right)$. We then obtain

$$\Delta v_{hc} = \frac{m}{R_{sph} \cos B_{th}} \left[45^\circ - \frac{180^\circ}{\pi} c_c - \frac{\pi^2 \, 15^\circ}{8 Z_c^2} \right] \quad \dots\dots\dots\dots \quad (272)$$

where Δv_{hc} is measured in degrees, c_c is taken from Appendix 8 for the selected milling cutter and Z_c is the theoretical value obtained from eq. (254), not the value for the selected milling cutter.

If, instead, the value of Z_c for the selected milling cutter is used, the maximum error will occur if Z_c obtained from eq. (254) is just below 14 and a cutter designed for $Z_c = 12$ teeth is used. The value of Δv_{hc} can then become too low, resulting in a tooth space which is 0.08 per cent too narrow. This deviation may be neglected in most cases.

In the foregoing derivations one uncontrolled approximation has been made, namely in the expression for ϱ_{qc}. We have chosen a value of ϱ_{qc} determined by

$$\varrho_{qc} = \frac{m Z_c}{2} \left(1 - \frac{2 F}{R_{sph} Z_c} \right)$$

which does not give quite the desired value of ϱ_{qc}. It will now be shown that the error originating from this approximation may be neglected.

The depth of the point a at the large end of the tooth space (see Fig. 22) measured vertically from the highest point of the pitch curve at the large end of the tooth should be

$$= \frac{m\, Z_{osc}}{2}\left(1 - \cos\frac{\pi}{2\, Z_{osc}}\right)\cos\left(\varrho_p - B_{tk}\right)$$

but, as can be understood from Fig. 20, it becomes

$$= m\,\frac{Z_c}{2}\left(1 - \cos\frac{\pi}{2\, Z_c}\right)$$

if the cutter is set to such a depth that at the large end of the teeth the highest points of the pitch curves for the wheel and for the cutter coincide.

Thus the point a will be too deeply cut a vertical distance

$$\delta_p = m\left[\frac{Z_c}{2}\left(1 - \cos\frac{\pi}{2\, Z_c}\right) - \frac{Z_{osc}}{2}\left(1 - \cos\frac{\pi}{2\, Z_{osc}}\right)\cos\left(\varrho_p - B_{tk}\right)\right]$$

If the expressions $1 - \cos\dfrac{\pi}{2\, Z_c}$ and $1 - \cos\dfrac{\pi}{2\, Z_{osc}}$ are expanded in power series, we obtain

$$\delta_p \approx m\left[\frac{\pi^2}{16\, Z_c} - \frac{\pi^2\cos\left(\varrho_p - B_{tk}\right)}{16\, Z_{osc}}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (273)$$

Z_c is here the value for the selected milling cutter, not the theoretical value from eq. (254).

If $Z_c = \dfrac{Z_{osc}}{\cos\left(\varrho_p - B_{tk}\right)}$, then $\delta_p = 0$, but even if we use whole numbers of cutters only, the maximum value of δ_p will be less than

$$m\,\frac{\pi^2}{16}\left[\frac{1}{12} - \frac{1}{14}\right] = m\ 0.0073$$

and may therefore be neglected. But, if so desired, it can be compensated for by setting the cutter that much higher.

For the point b at the small end of the tooth space (see Fig. 22) we obtain in a similar way the desired depth

$$= m\,\frac{F}{R_{sph}} + m\,\frac{Z_{osc}}{2}\left(1 - \frac{F}{R_{sph}}\right)\left(1 - \cos\frac{\pi}{2\, Z_{osc}}\right)\cos\left(\varrho_p - B_{tk}\right)$$

and the real depth (see Fig. 20)

$$= \frac{m\, Z_c}{2} - \frac{m\, Z_c}{2}\left(1 - \frac{2\, F}{R_{sph}\, Z_c}\right)\cos\left(\sigma_{bc} + \beta_s\right)$$

$$= m\,\frac{F}{R_{sph}} + m\left[\frac{Z_c}{2} - \frac{F}{R_{sph}}\right]\left[1 - \cos\left(\sigma_{bc} + \beta_s\right)\right]$$

Thus the point b is cut too deeply a vertical distance δ_q, where

$$\delta_q = m\left\{\left(\frac{Z_c}{2} - \frac{F}{R_{sph}}\right)\left[1 - \cos\left(\sigma_{bc} + \beta_s\right)\right] - \frac{Z_{osc}}{2}\left(1 - \frac{F}{R_{sph}}\right)\left(1 - \cos\frac{\pi}{2\, Z_{osc}}\right)\cos\left(\varrho_p - B_{tk}\right)\right\}$$

The angle $(\sigma_{bc} + \beta_s)$ may be determined in the following way. From Figs. 20 and 21 we have

$$k_{pc} - k_{qc} = c_c\, m \frac{F}{R_{sph}} = \varrho_{pc} \sin \frac{\pi}{2\, Z_c} - \varrho_{qc} \sin (\sigma_{bc} + \beta_s) =$$

$$= \frac{m\, Z_c}{2} \sin \frac{\pi}{2\, Z_c} - m \left(\frac{Z_c}{2} - \frac{F}{R_{sph}} \right) \sin (\sigma_{bc} + \beta_s)$$

from which

$$\sin (\sigma_{bc} + \beta_s) = \frac{\sin \frac{\pi}{2\, Z_c} - \frac{2\, c_c}{Z_c} \cdot \frac{F}{R_{sph}}}{1 - \frac{2}{Z_c} \cdot \frac{F}{R_{sph}}} \approx \frac{\frac{\pi}{2\, Z_c} - \frac{\pi^3}{48\, Z_c^3} - \frac{2\, c_c}{Z_c} \cdot \frac{F}{R_{sph}}}{1 - \frac{2}{Z_c} \cdot \frac{F}{R_{sph}}} \approx$$

$$\approx \frac{\pi}{2\, Z_c} - \frac{\pi^3}{48\, Z_c^3} - \left(\frac{2\, c_c}{Z_c} - \frac{\pi}{Z_c^2} \right) \frac{F}{R_{sph}} - \left(\frac{4\, c_c}{Z_c^2} - \frac{2\, \pi}{Z_c^3} \right) \left(\frac{F}{R_{sph}} \right)^2$$

and

$$(\sigma_{bc} + \beta_s) \approx \frac{\pi}{2\, Z_c} - \left(\frac{2\, c_c}{Z_c} - \frac{\pi}{Z_c^2} \right) \frac{F}{R_{sph}} - \frac{4\, c_c}{Z_c^2} \left(\frac{F}{R_{sph}} \right)^2$$

From this last expression we now form

$$[1 - \cos (\sigma_{bc} + \beta_s)] \approx \frac{\pi^2}{8\, Z_c^2} - \left(\frac{c_c\, \pi}{Z_c^2} - \frac{\pi^2}{2\, Z_c^3} \right) \frac{F}{R_{sph}} + \left(\frac{2\, c_c^2}{Z_c^2} - \frac{4\, \pi\, c_c}{Z_c^3} \right) \left(\frac{F}{R_{sph}} \right)^2$$

Further we have

$$\left[1 - \cos \frac{\pi}{2\, Z_{osc}} \right] \approx \frac{\pi^2}{8\, Z_{osc}^2} - \frac{\pi^4}{384\, Z_{osc}^4}$$

If these expressions are inserted in the above expression for δ_q, we obtain

$$\delta_q = m \left[\frac{\pi^2}{16\, Z_c} - \frac{\pi^2 \cos (\varrho_p - B_{th})}{16\, Z_{osc}} + \left(\frac{\pi^2 \cos (\varrho_p - B_{th})}{16\, Z_{osc}} - \frac{c_c\, \pi}{2\, Z_c} + \frac{\pi^2}{8\, Z_c^2} \cdots \right) \frac{F}{R_{sph}} + \right.$$

$$\left. + \left(\frac{c_c^2}{Z_c} - \frac{c_c\, \pi}{Z_c^2} \right) \left(\frac{F}{R_{sph}} \right)^2 \cdots \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (274)$$

It can be shown that the largest value to which δ_q can amount for a pressure angle of 20° is $m \cdot 0.009$, which occurs for $\frac{Z_{osc}}{\cos (\varrho_p - B_{th})}$ just below 14, $Z_c = 12$ and $\frac{F}{R_{sph}} = \frac{1}{3}$, giving $c_c = 0.376$.

For a pressure angle of 20°, the error δ_q can thus be neglected, especially if δ_p is compensated for by setting the cutter that much higher. The remaining error in δ_q will then never exceed $m \cdot 0.004$ (for $Z_c = 12$). The error δ_q means that the correct depth is obtained at a slight distance from the small end of the tooth. Since the generatrix becomes somewhat concave, this may even be regarded as an advantage.

If so desired, the two errors δ_p and δ_q can both be compensated for by decreasing the cutting angle, i.e. by increasing the angle B_{th} in the system of equations (259) by an amount ΔB_{th} and at the same time decreasing the value of h_c obtained from eqs. (265) by an amount Δh_c obtained from

$$\left. \begin{aligned} \Delta B_{th} &= \frac{\delta_q - \delta_p}{F} \\ \Delta h_c &= \delta_p \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (275)$$

The above derivations of the milling-machine settings are in the first place valid if the apex of the pitch cone coincides with the centre of the dividing head. The value of b_{so} then gives directly the cross-setting of the work table. With the normal set-up as shown in Figs. 13 and 14, Δv_{hc} from eq. (272) will still be the change from the value of v_{hc} which is valid for the central cut. The position, however, of the apex of the pitch cone is changed by altering the setting v_{hc} to $v_{hc} \pm \Delta v_{hc}$. Therefore, b_{so} is the change from the new value of the cross-setting, determined in the same way as the corresponding value for the central cut, although with v_{hc} replaced by $v_{hc} \pm \Delta v_{hc}$ in eqs. (265).

For the side cuts we therefore obtain

$$\left.\begin{aligned} v_{hcL} &= v_{hc} - \Delta v_{hc} \\ l_{cL} &= f \sin (v_{hc} - \Delta v_{hc}) \cos v_{vc} + b_{so} \\ v_{hcR} &= v_{hc} + \Delta v_{hc} \\ l_{cR} &= f \sin (v_{hc} + \Delta v_{hc}) \cos v_{vc} - b_{so} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (276)$$

where

L = a suffix indicating the left-hand side cuts

R = " " " " right-hand " "

Appendix 10 gives a compilation of the formulae for calculating the settings when milling non-circular bevel gear wheels.

2) *Circular bevel gear wheels*

Although these investigations mainly concern non-circular bevel gears, it might be advantageous to treat also the cutting of the teeth of circular bevel gears here.

For circular bevel gears the dividing head is not usually erected on a rotary milling table, but directly on the milling-machine table. When cutting the side cuts, the arbor of the dividing head is then rotated round its axis in order to obtain the correct tooth width.

The number of teeth for which the cutter should be designed is determined from eq. (254) which, since in this case $\frac{m Z}{2} = R_{sph} \sin \alpha_p$ and $\varrho_p = \alpha_p$, where α_p is half the apex angle of the pitch cone, may be written in the form

$$Z_{cC} = \frac{Z}{\cos \alpha_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (277)$$

The suffix C indicates that the value belongs to a circular bevel wheel.

A standard cutter for the next lower number of teeth than that calculated from the above equation should be chosen.

Half the tooth space must cover an angle on the pitch circle equal to $\frac{\pi}{2Z}$. The out-of-centre setting is equivalent to an angle v_z determined by (see Figs. 21 and 23)

$$\sin v_z = \frac{c_c m}{m \frac{Z}{2}} = \frac{2 c_c}{Z}$$

or

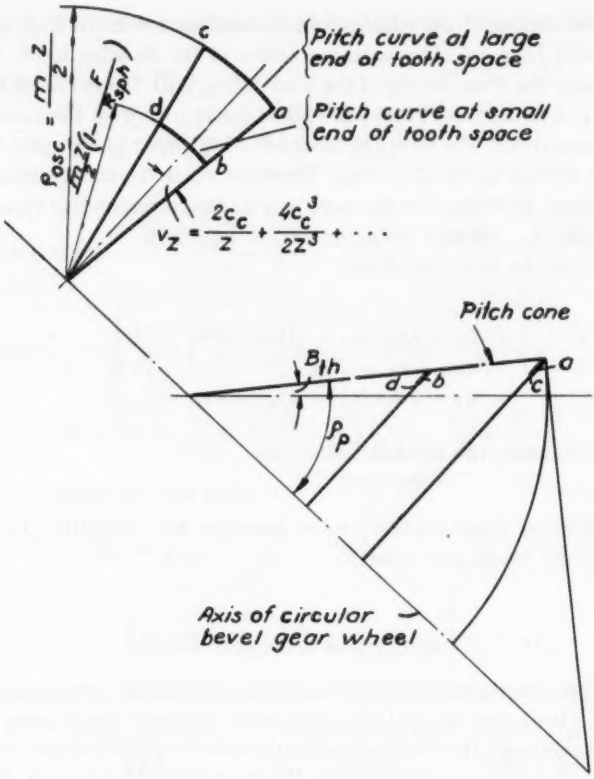

Pitch curve at large end of tooth space
Pitch curve at small end of tooth space
Pitch cone
Axis of circular bevel gear wheel

Fig. 23. Sketch for determining the desired depth below the highest point of the pitch curve for points *a* and *b*. The sketch applies to circular bevel gear wheels where no rotary milling table is used, but the correct width of the tooth space is obtained by rotating the arbor of the dividing head.

$$v_z = \frac{2\,c_c}{Z} + \frac{4\,c_c^3}{3\,Z^3} \cdots$$

When cutting the side cuts, the arbor of the dividing head must, therefore, be rotated through an angle

$$\Delta\, v_{dC} = \pm \frac{\pi}{2\,Z} \left(1 - \frac{4\,c_c}{\pi} - \frac{8\,c_c^3}{3\,\pi\,Z^2} \cdots \right)$$

As $c_c < 0.44$ and $Z \geq 12$, the value of the last term is less than 0.004° and may therefore be neglected. We thus obtain

$$\Delta\, v_{dC} = \pm \frac{90^\circ}{Z} \left(1 - \frac{4\,c_c}{\pi} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (278)$$

where $\Delta\, v_{dC}$ is measured in degrees.

The blank is rotated in such a direction that the width of the tooth space is increased.

This additional rotation is most easily obtained by using a dividing head having, in addition to the ordinary index plate, differential indexing. The central cuts are then at first milled with the differential index plate set at zero. When milling the side cuts, the differential indexing is set at the value Δv_{4C} in the desired direction. Fig. 24 shows the arrangement. The differential index plate should be graduated in degrees.

The out-of-centre setting of the cutter for circular bevel gears may also be determined from Appendix 8.

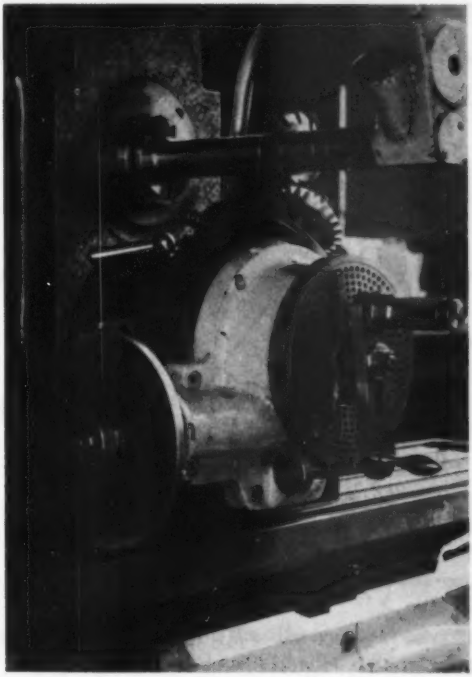


Fig. 24. Cutting circular bevel gear wheel with disc milling cutter. A dividing head is used having a differential setting plate, seen to the left, in addition to the ordinary indexing plate.

The two points where the tooth profile should intersect the pitch circles at the large end and the small end of the teeth must still fall on the same line as the apex of the pitch cone. If, however, the blank is rotated around its axis, these two points will be cut too deeply.

After the wheel has been rotated to the position for cutting, then the depth of point a at the large end of the tooth space (see Fig. 23) measured vertically from the highest point of the pitch curve at the large end of the tooth space should be

$$= m \frac{Z}{2} \left[1 - \cos \left(\frac{2 c_c}{Z} + \frac{4 c_c^3}{3 Z^3} \dots \right) \right] \cos (\varrho_p - B_{tk})$$

but, as can be seen from Fig. 20, it becomes

$$= m \frac{Z_c}{2} \left(1 - \cos \frac{\pi}{2 Z} \right)$$

if the cutter is set to such a depth that the highest points of the pitch circles for the wheel and the cutter coincide. Thus the point a is cut too deeply a distance

$$\delta_{pC} = m \left\{ \frac{Z_c}{2} \left(1 - \cos \frac{\pi}{2 Z_c} \right) - \frac{Z}{2} \left[1 - \cos \left(\frac{2 c_c}{Z} + \frac{4}{3} \frac{c_c^3}{Z^3} \cdots \right) \right] \cos (\varrho_p - B_{th}) \right\}$$

If $\left(1 - \cos \frac{\pi}{2 Z_c} \right)$ and $\left[1 - \cos \left(\frac{2 c_c}{Z} + \frac{4}{3} \frac{c_c^3}{Z^3} \cdots \right) \right]$ are expanded in power series, the expression takes the form

$$\delta_{pC} = m \left| \frac{\pi^2}{16 Z_c} - \frac{c_c^2}{Z} \cos (\varrho_p - B_{th}) + \cdots \right]$$

For the point b at the small end of the tooth space we obtain from Fig. 23 the desired depth

$$= m \frac{F}{R_{sph}} + m \frac{Z}{2} \left[1 - \cos \left(\frac{2 c_c}{Z} + \frac{4}{3} \frac{c_c}{Z^3} \cdots \right) \right] \left(1 - \frac{F}{R_{sph}} \right) \cos (\varrho_p - B_{th})$$

and the real depth from Fig. 20

$$= m \frac{F}{R_{sph}} + m \frac{Z_c}{2} \left[1 - \cos (\sigma_{bc} + \beta_s) \right] \left(1 - \frac{2 F}{R_{sph} Z_c} \right)$$

Thus point b is cut too deeply a distance

$$\delta_{qC} = m \left\{ \frac{Z_c}{2} \left[1 - \cos (\sigma_{bc} + \beta_s) \right] \left(1 - \frac{2 F}{R_{sph} Z_c} \right) - \right.$$

$$\left. - \frac{Z}{2} \left[1 - \cos \left(\frac{2 c_c}{Z} + \frac{4}{3} \frac{c_c^3}{Z^3} \cdots \right) \right] \left(1 - \frac{F}{R_{sph}} \right) \cos (\varrho_p - B_{th}) \right\}$$

If the series expression of $[1 - \cos (\sigma_{bc} + \beta_s)]$ from previous derivations for non-circular bevel wheels on page 138 as also the series expression for $\left[1 - \cos \left(\frac{2 c_c}{Z} + \frac{4}{3} \frac{c_c^3}{Z^3} \cdots \right) \right]$ are inserted, we obtain

$$\delta_{qC} \approx m \left\{ \frac{\pi^2}{16 Z_c} - \left(\frac{c_c \pi}{2 Z_c} - \frac{\pi^2}{8 Z_c^2} \right) \frac{F}{R_{sph}} - \frac{c_c^2}{Z} \cos (\varrho_p - B_{th}) + \frac{c_c^2}{Z} \cdot \frac{F}{R_{sph}} \cos (\varrho_p - B_{th}) \right\}$$

The two errors δ_{pC} and δ_{qC} can both be compensated for by setting the cutter to too high a distance δ_{pC}, and by increasing the cutting angle by an amount

$$\Delta V = \frac{\delta_{pC} - \delta_{qC}}{F} = \frac{m}{R_{sph}} \left[\frac{c_c \pi}{2 Z_c} - \frac{c_c^2}{Z} \cos (\varrho_p - B_{th}) - \frac{\pi^2}{8 Z_c^2} \right]$$

To make the theoretical bottom of the tooth space horizontal, the cutting angle should be chosen as

$$\varrho_p - \text{arc sin} \frac{m}{R_{sph}}$$

In the following we use for circular bevel gears the notations

α_p = half the apex angle of the pitch cone

V = the cutting angle

Thus $\alpha_p = \varrho_p$

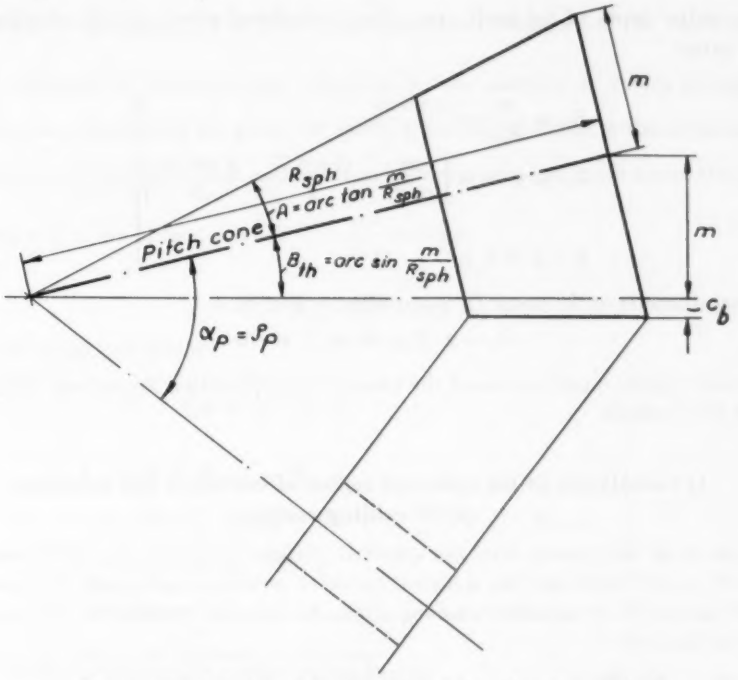


Fig. 25. Usual design of the teeth for circular bevel gear wheels.

Using these new notations and neglecting the difference between $\cos \alpha_p$ and $\cos (\alpha_p - B_{th})$ and the last term in ΔV, the cutting angle will now be determined by

$$\left.\begin{aligned} V &= \alpha_p - B_{th} + \Delta V \\ \sin B_{th} &= \frac{m}{R_{sph}} = \frac{2 \sin \alpha_p}{Z} \\ \Delta V &= \frac{m}{R_{sph}} \left(\frac{c_c \pi}{2 Z_c} - \frac{c_c^2}{Z} \cos \alpha_p \right) \frac{180^\circ}{\pi} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (279)$$

where

where the values are expressed in degrees.

In order to obtain the correct depth of the tooth space, the elevation of the milling-machine table is then adjusted so that the cutter touches the highest point on the contour of the blank. After this, the table is elevated a distance h_{cC}, so chosen that the depth of the tooth space will be correct. If the wheel is designed according to Fig. 25, this distance comprises the distances $m + c_b$ and the height $m \cos B_{th}$ of the tooth tip above the pitch curve less the distance δ_{pC}.

Thus we have

$$h_{cC} = m + c_b + m \cos B_{th} - \delta_{pC}$$

and

$$2\,m + c_b = d$$

where

d = the entire depth of the tooth space for a cylindrical wheel, usually stamped on the milling cutter

or

$$\left.\begin{aligned} B_{th} &\approx \frac{m}{R_{sph}} = \frac{2 \sin \alpha_p}{Z} \\ h_{cC} &\approx d - m \left[\frac{\pi^2}{16\, Z_c} - \frac{c_c^2 \cos V}{Z} + \frac{2 \sin^2 \alpha_p}{Z^2} \right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(280)}$$

where

$$d = 2\,m + c_b$$

The outer diameter of the blank D_y is according to Fig. 25

$$D_y = 2\,[R_{sph} \sin \alpha_p + m \cos \alpha_p] \ldots\ldots\ldots\ldots\ldots\ldots \text{(281)}$$

Appendix 9 gives a compilation of the formulae for calculating the settings when milling circular bevel wheels.

f) Calculation of the spherical radius of curvature and selection of the milling cutters

Fig. 26 shows with heavy lines the spherical triangle formed by the radius vector r to a point P_1 on the pitch line, the spherical radius of curvature and an arc of a great circle through the centre of curvature and the origin. An adjacent triangle for the point P_2 is shown by fine lines.

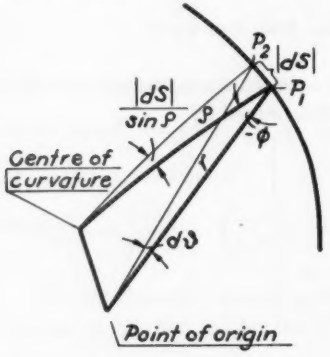


Fig. 26. From the differential law for spherical triangles
$d\alpha = -\cos b \cdot d\gamma - \cos c \cdot d\beta + \sin \beta \sin c \cdot da$
can be derived the formula
$\tan \varrho = \dfrac{|dS|}{d\Phi + \cos r \cdot d\vartheta}$

A well-known differential law for spherical triangles states that

$$d\alpha = -\cos b \cdot d\gamma - \cos c \cdot d\beta + \sin \beta \sin c \cdot d\,a$$

where α, β and γ are the angles of the triangle and a, b and c their opposite sides.

If a is a constant, the law takes the form

$$d\alpha = -\cos b \cdot d\gamma - \cos c \cdot d\beta$$

If this law is applied to the spherical triangle in Fig. 26, where the side formed by the great circle through the origin and the centre of curvature for the pitch curve is constant, we obtain

$$-d\,\Phi = \cos r \cdot d\vartheta - \cos \varrho \frac{|dS|}{\sin \varrho}$$

from which

$$\tan \varrho = \frac{|dS|}{d\Phi + d\vartheta \cos r} \ldots\ldots\ldots \text{(282)}$$

An approximate expression for the radius of curvature is obtained if in eq. (282) we replace the differentials $|dS|$, $d\Phi$ and $d\vartheta$ by the differences ΔS, $\Delta\Phi$ and $\Delta\vartheta$ between two adjacent points, for example, the two points on the pitch curve lying in the centre of two adjacent teeth. For r we take the value between these two points, i.e. in the centre of the intermediate tooth space. A still better approximation is obtained if the values of $\frac{d\Phi}{|dS|}$ and $\frac{d\vartheta}{|dS|}$ obtained by second degree interpolation are inserted. If values belonging to the tooth space in question are given the suffix n, values belonging to the adjacent teeth the suffixes $n-\frac{1}{2}$ and $n+\frac{1}{2}$ and values belonging to the adjacent tooth spaces the suffixes $n-1$ and $n+1$, we obtain, since $|dS| = \frac{m\pi}{R_{sph}}$ radians,

$$\tan \varrho_n \approx \frac{m\pi}{R_{sph}\,[\Delta\Phi + \Delta\vartheta \cos (r)_n]}$$

Using eq. (254), we obtain

$$(Z_c)_n \approx \frac{360^\circ}{\Delta\Phi + \Delta\vartheta \cos (r)_n}$$

where

$$\left.\begin{aligned} \Delta\Phi &= \Phi_{n+\frac{1}{2}} - \Phi_{n-\frac{1}{2}} \\ \text{or } \Delta\Phi &= \frac{4}{3}\,(\Phi_{n+\frac{1}{2}} - \Phi_{n-\frac{1}{2}}) - \frac{1}{6}\,(\Phi_{n+1} - \Phi_{n-1}) \\ \Delta\vartheta &= \vartheta_{n+\frac{1}{2}} - \vartheta_{n-\frac{1}{2}} \\ \text{or } \Delta\vartheta &= \frac{4}{3}\,(\vartheta_{n+\frac{1}{2}} - \vartheta_{n-\frac{1}{2}}) - \frac{1}{6}\,(\vartheta_{n+1} - \vartheta_{n-1}) \end{aligned}\right\} \quad \ldots\ldots\ldots (283)$$

and where the angles are expressed in degrees.

The second expressions for $\Delta\Phi$ and $\Delta\vartheta$ are in fact the values of $\frac{m\pi}{R_{sph}} \cdot \frac{d\Phi}{|dS|}$ and $\frac{m\pi}{R_{sph}} \cdot \frac{d\vartheta}{|dS|}$ obtained by second degree interpolation. By using these values, it is possible to obtain a more precise value of $(Z_c)_n$.

A standard cutter for the next lower number of teeth than that calculated from the above equations should be chosen. Standard milling cutters are available for the number of teeth given in the table of Appendix 8.

If Z_c happens to be negative, this means that the pitch curve is concave and that a cutter for internal teeth must be used; in this case also with the next algebraically lower number of teeth than that calculated.

B. Machining the contour of the blanks

The machining of the contour of the blanks is carried out in a similar way to the tooth cutting. From a chronological point of view the machining of the blanks ought to have been treated before the tooth cutting. By treating the tooth cutting, however, a series of formulae have been obtained which are also required for the description of the machining of the contour of the blanks. This is the reason why this process has not been described until now.

The blank is set up in the milling machine in the same way as it must be erected later when cutting the teeth, i.e. the rotary milling table is erected on the milling-machine table, and the dividing head is in turn placed on this in such a way that the vertical axes of both coincide. Instead of the milling cutter, however, a cylindrical cutter is used, having about the same length as the tooth pitch. The blank is adjusted by means of the setting levers of the dividing head and the rotary milling table so that the generatrix at a tooth tip becomes horizontal and assumes the same direction as the longitudinal axis of the milling-machine table. The height- and cross-settings are adjusted so that the cutter comes directly above the tooth tip and cuts the tooth tip to the correct height. This procedure is repeated for all the teeth round the whole periphery. The blank becomes indeed angular (see Fig. 31), but this is of no consequence since the edges are cut away when cutting the teeth. The tooth tips will also become plane, but this also is of no importance and may eventually be avoided by using a somewhat concave cutter having a radius of the profile approximately equal to the radius of curvature of the desired profile.

The equations (259) may be used after some modification for determining the machine settings. With settings obtained from these equations, the theoretical bottom of the tooth space will lie horizontal, but here the generatrix at the tooth tip must be horizontal. This gives formally the same equations, although B_{th} must be replaced by the negative value of A from eq. (261), and further the values of r_c and Φ_c must be referred to the tooth tip instead of to the bottom of the tooth space. If the suffix t indicates values belonging to the tooth tip, then equations (259) change to

$$\left.\begin{aligned}
&\cos r_t = \cos A \cos r - \sin A \sin r \cos \Phi \\
&\sin \Phi_t = \frac{\sin r}{\sin r_t} \sin \Phi \\
&\sin v_{vt} = \sin r_t \cos \Phi_t \\
&\tan v_{ht} = \tan r_t \sin \Phi_t \\
&\sin (\Theta_t - \Theta) = \frac{\sin A}{\sin r} \sin \Phi_t \\
&\tan (v_{dt} - \Theta_t) = \frac{1}{\cos r_t} \tan \Phi_t \\
&v_{dt} = (v_{dt} - \Theta_t) + (\Theta_t - \Theta) + \Theta
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (284)$$

A may be obtained from one of the equations (261). If the first one is used, we obtain $A = B_{th}$.

To obtain the correct height setting, we may place on the arbor of the dividing head a setting disc having a diameter $= D_t$. The arbor of the dividing head is set parallel with the milling-machine table and the cross-setting of the table is adjusted so that the cutter comes directly above the centre of the arbor and touches the setting disc, after which the scales are set at zero and locked. The height settings h_t and cross-settings l_t of the milling-machine table when milling the tips of the teeth will now be determined by [compare eqs. (265)]

$$\left.\begin{aligned}
h_t &= \frac{D_t}{2} - f \sin v_{vt} \\
l_t &= f \cos v_{vt} \sin v_{ht}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (285)$$

where

D_t = the diameter of the setting disc

h_t = the height setting of the milling-machine table

l_t = the cross-setting of the milling-machine table

If a setting disc is used exclusively for the milling of the tips of the teeth, then its diameter is conveniently chosen as

$$D_t = 2\,f \sin (v_{vt})_{\max} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (286)$$

The lowest value of h_t will then equal zero. The diameter of the setting disc may, however, be chosen arbitrarily, for example, the same setting disc as for the tooth cutting may be used, and eqs. (285) will still be valid.

The above formulae are compiled in Appendix 10.

C. Preparation of cutting schedules for cutting the tooth spaces and the contour of the blanks (the tips of the teeth)

When all values necessary for the machine settings have been calculated, they should be compiled in one cutting table for the milling of the contour of the blank, and another table for the cutting of the teeth convenient for the use of the milling-machine operator. Appendices 19, 20 and 21 show examples of such schedules. In these schedules the numbering of the teeth and the tooth spaces should be made continuous. If, when designing the gear, the pitch curve has been divided into different arcs, the numbering in the cutting schedules will then not agree with the numbering used above, where it was applied separately for each arc.

For multi-period wheels where each period comprises an integral number of teeth, the calculations need only be carried out for one period. The values for the other periods may be obtained for reasons of symmetry. For each following period on a n-period wheel the values of v_{dc} and v_{dt} must be increased by an amount equal to $\frac{360^\circ}{n}$. All other values will remain unchanged. Further, if the wheel has an axis of symmetry, only one half of the period will have to be evaluated, the value for the other half being easily obtained for reasons of symmetry. In the cutting schedules, however, the settings for all tooth spaces and tooth tips should be listed.

Previously, it has been assumed that the different milling cutters used have exactly the same outer diameter and the same length of the boss, and that the central planes of the profiles lie exactly in the centre of the cutters. If this should not be the case, the machine settings will have to be modified accordingly.

As a rule, dividing heads have a worm-gear reduction for the indexing but not for the setting of the tilting of the arbor. This last-mentioned setting therefore does not give quite sufficient accuracy. If, for example, the setting can be read at an accuracy of $\pm 0.05^\circ$ and the distance from the centre of the dividing head to the teeth is 100 mm, the error in

the depth of the tooth space can amount to $\frac{0.05\,\pi}{180}\,100 = 0.09$ mm, which is too much for more accurate work. It would therefore be desirable to have a worm-gear reduction also for the tilting of the arbor.

D. Examples of the manufacture of non-circular bevel gear wheels

a) A bevel gear the pitch curves of which are a spherical ellipse and the two-period rolling curve to it

1) Determination of the spherical pitch curves for a given gear ratio and a given angle between the shafts

The calculation of a non-circular bevel gear train will be illustrated by a very simple numerical example according to the following conditions.

Mean gear ratio 2/1

Min. instantaneous ratio $|u_{L/R}|_{\min} = 0.9$ to 1.1

Max. instantaneous ratio $|u_{L/R}|_{\max} = 4.0$ to 4.3

Shaft angle $|C_{L-R}| = 90°$

Apex distance $R_{sph} = 65$ to 70 mm

Module = 3

The wheels should be symmetrical

No special stipulations are given for the shape of the gear ratio curve between its max. and min. values.

Preliminarily we choose

$|u_{L/R}|_{\min} = 1.00$ corresponding to the initial position and

$|u_{L/R}|_{\max} = 4.15$ corresponding to the end position of the arc to be computed

According to eq. (249) we now have

$$\tan R_{sL} = \frac{\sin 90°}{\cos 90° + 1.00} = 1.00 \text{ or } R_{sL} = 45.000°$$

$$\tan R_{eL} = \frac{\sin 90°}{\cos 90° + 4.15} = 0.24096 \text{ or } R_{eL} = 13.547°$$

As the wheels are symmetrical, we have also for the L-wheel

$$\tan \Phi_s = 0,\ \tan \Phi_e = 0 \text{ and } \Delta\Theta = 180°$$

The preliminary L-curve may now be determined by following Appendix 2. We obtain

$$\tan \bar{\Phi}_s = \tan \Phi_s = 0,\ \tan \bar{\Phi}_e = \tan \Phi_e = 0$$

$$k_3 = \frac{180°}{2} \cdot \frac{\tan R_e \tan \bar{\Phi}_s + \tan R_s \tan \bar{\Phi}_e}{\tan R_s - \tan R_e} = 0$$

and

$$\tan\left(\nu\frac{180^\circ}{2}\right)=\infty, \text{ from which } \nu=1$$

With the aid of Appendix 2 we further determine

$$k_4=0 \text{ and } \tan\nu\vartheta_s=0, \text{ from which } \vartheta_s=0$$

$$k_2=\frac{\tan R_s+\tan R_e}{\tan R_s-\tan R_e}=\frac{1.24096}{0.75904}=1.63491$$

$$k_1=\frac{\tan R_s\tan R_e}{\tan R_s-\tan R_e}\cdot 2=\frac{2\cdot 0.24096}{0.75904}=0.63491$$

The equation of the preliminary L-curve can now be written in the form

$$\tan r_L=\frac{k_1}{k_2-\cos\vartheta_L}=\frac{0.63491}{1.63491-\cos\vartheta_L}$$

Using Appendix 2 we further determine

$$\tan 2a_1=\frac{2k_1k_2}{k_2^2-k_1^2-1}=\frac{2\cdot 1.63491\cdot 0.63491}{2.67293-0.40311-1}=1.63491, \text{ from which } a_1=29.274^\circ$$

$$\sin\alpha_T=\frac{2k_2}{k_2^2+k_1^2+1}=\frac{2\cdot 1.63491}{2.67293+0.40311+1}=0.80221, \text{ from which } \alpha_T=53.341^\circ$$

$$\sin\alpha_1=\tan\frac{\alpha_T}{2}=\tan 26.671^\circ=0.50231, \text{ from which } \alpha_1=30.153^\circ$$

The L-curve is thus a real spherical conic section having $\alpha_1=30.153^\circ$ and $a_1=29.274^\circ$. For the main curve r_L lies in the first quadrant and for the antipode curve in the third quadrant. Thus, for external meshing and $|C_{L-R}|=90^\circ$, C_{L-R} is positive and the required substitute curve is the main curve.

The mating curve corresponding to a shaft angle $C_{L-R}=90^\circ$ may now be determined by using Appendices 3 and 4. We obtain from Appendix 3

$$C_{L-R}=a_{1L}+a_{\nu R}=a_1(1+\varkappa_R)=29.274\ (1+\varkappa_R)=90^\circ$$

from which $\varkappa_R=2.074$ and $a_{\nu R}=\varkappa_R a_1=60.726^\circ$. Both curves are main curves.

With the aid of Appendix 4 we obtain also from eq. (108)

$$\sin\alpha_{\nu R}=\frac{\tan a_1\sin\alpha_1}{\tan a_{\nu R}}=\frac{0.56058\cdot 0.50231}{1.78388}=0.15785$$

from which $\alpha_{\nu R}=9.082^\circ$

$$\nu_R=\frac{\sin a_{\nu R}\cos\alpha_{\nu R}}{\sin a_1\cos\alpha_1}=\frac{0.87229\cdot 0.98746}{0.48899\cdot 0.86468}=2.03716$$

According to Appendix 6 the angle of displacement of the R-wheel when the L-wheel rotates from $\vartheta_L = 0$ to $\vartheta_L = 180°$ equals $-\frac{180°}{\nu_R}$. The curves preliminarily obtained must therefore be adjusted by multiplying the gear ratio by the constant c given by eq. (251).

$$c = \frac{2}{360°}\, 2\, \frac{180°}{2.03716} = \frac{1}{1.01858}$$

The L-curve must be recalculated with the ratios changed in the proportion c. By using eq. (252) this is very simple. The new values of k_4 and ν will remain unaltered. If eq. (250) is used for the determination of the new values of k_1 and k_2, we now find that k_1 will increase in the proportion $\frac{1}{c}$, i.e. the new value of k_1 becomes $1.01858 \cdot 0.63491 = 0.64671$. As $\tan C_{L-R} = \infty$, then according to eq. (250) the new value of k_2 will remain unchanged. The equation of the adjusted L-curve thus becomes

$$\tan r_L = \frac{0.64671}{1.63491 - \cos \vartheta_L}$$

If also the constants α_1 and a_1 are evaluated for this new curve, we obtain by using Appendix 2

$$a_1 = 29.659° \text{ and } \alpha_{rL} = \alpha_1 = 29.949°$$

For the mating curve we obtain with the aid of Appendices 3 and 4

$$a_{rR} = 60.341°,\ \alpha_{rR} = 9.316° \text{ and } \nu_R = 2.00000$$

As pitch curves, we have thus obtained a spherical rolling curve pair, consisting of a spherical conic section and a two-period rolling curve to this, with $\alpha_1 = 29.949°$, i.e. almost equal to 30°. An appreciable simplification of the calculations, especially the determination of the arc length of the curves, could, however, be attained by choosing α_1 exactly equal to 30°. Obviously such a choice would not appreciably affect the gear ratio curve. As definite value we therefore choose an angle of eccentricity $\alpha_1 = 30°$. Starting with this value of α_1, the given shaft angle 90° and the given mean gear ratio 2/1, corresponding to $\nu_L = 1$ and $\nu_R = 2$, the definite rolling curves can be determined. This has already been carried out in the example in Chapter V : A a 1, on pages 48—50, where the following results were obtained

$$a_1 = 29.670°,\ \alpha_1 = 30°,\ a_2 = \varkappa_2\, a_1 = 60.330°,\ \alpha_2 = 9.339°$$

and the equations of the rolling curves are in parametric form, as follows
The L-curve:

$$\left.\begin{aligned} \tan (r_1 - 29.670°) &= \tan 29.670° \sin 30° \sin \varphi_1 = 0.28485 \sin \varphi_1 \\ \tan \frac{\vartheta_1}{2} &= \tan \frac{90° - 30°}{2} \tan \frac{90° - \varphi_1}{2} = 0.57735 \tan \frac{90° - \varphi_1}{2} \end{aligned}\right\} \quad \ldots\ldots \quad (287)$$

The R-curve:

$$\left.\begin{aligned} \tan (r_2 - 60.330°) &= \tan 29.670° \sin 30° \sin \varphi_2 = 0.28485 \sin \varphi_2 \\ \tan \vartheta_2 &= \tan \frac{90° - 9.339°}{2} \tan \frac{90° - \varphi_2}{2} = 0.84898 \tan \frac{90° - \varphi_2}{2} \end{aligned}\right\} \quad \ldots\ldots \quad (288)$$

These two curves are those shown in Figs. 6 and 7a and are chosen as the definite pitch curves.

As we have external meshing, $\varphi_R = -\varphi_L$, i.e. $\varphi_2 = -\varphi_1$ according to Appendix 3.

If we check the min. and max. gear ratios which appear at $\varphi_1 = \pm 90°$, we find by putting $\varphi_1 = \pm 90°$ in the first equation of (287),

$$\tan (r_1 - 29.670°) = \pm 0.28485$$

from which

$$(r_1)_{max} = 45.570°, \; (r_1)_{min} = 13.770°$$

Since the shaft angle $|C_{L-R}|$ is 90°, we have from eq. (6)

$$|u_{L/R}| = \frac{1}{\tan r_L}$$

Thus

$$|u_{L/R}|_{min} = \frac{1}{\tan (r_1)_{max}} = 0.9803$$

$$|u_{L/R}|_{max} = \frac{1}{\tan (r_1)_{min}} = 4.0805$$

These values are within the prescribed limits.

2) Calculation of φ, Φ, r and ϑ for the different teeth and tooth spaces

When calculating φ, Φ, r and ϑ for the midpoint of a tooth or tooth space, we must evaluate the arc length S_n to the midpoint of the tooth or tooth space in question. Firstly, a table is determined, giving the arc length S in radians as a function of φ, by using Simpson's formula, and the expressions for dS given in Appendix 7. For the curves concerned such a table has already been determined in Chapter IV:A d and will be found in Appendix 11.

From this table we find $S_{1/4} = 0.72741$. As according to the given conditions we must choose $R_{sph} = 65$ to 70 mm, the perimeter of the L-curve (the spherical ellipse) is $4 \cdot 0.72741 \cdot (65 \text{ to } 70) = 189.127$ to 203.675 mm. We may therefore choose

$$Z = 21 \text{ for the } L\text{-curve}$$

$$m = 3$$

corresponding to a perimeter of $3 \cdot \pi \cdot 21 = 197.920$ mm for the L-curve.

We further choose

$$R_{sph} = 68 \text{ mm}$$

making the real perimeter of the L-curve equal to $68 \cdot 4 \cdot S_{1/4} = 68 \cdot 4 \cdot 0.727414 = 197.857$ mm, if we use the somewhat more accurate value of $S_{1/4} = 0.727414$ evaluated in Chapter IV:A d on page 29. The real perimeter of the pitch curves will therefore be slightly less (0.032 per cent) than that corresponding to the module, which is quite suitable in order to ensure that the teeth will not become too wide.

The real tooth pitch ΔS now becomes

$$\Delta S = \frac{4 \cdot S_{1/4}}{Z} = \frac{4 \cdot 0.727414}{21} = 0.138555 \text{ radians}$$

or

$$R_{spk} \cdot 0.138555 = 68 \cdot 0.138555 = 9.422 \text{ mm}$$

instead of

$$m \cdot \pi = 9.425 \text{ mm}$$

corresponding to the module. The teeth will thus be only 0.003 mm too thin. The additional calculations are carried out in Appendix 13. By using the value of ΔS calculated above and the table giving the arc length (Appendix 11), we first evaluate S_n for all the teeth and tooth spaces for one half of the single-period L-wheel. A tooth space is placed at the max. radius. This tooth space thus corresponds to $\varphi = 90°$ and $S = 0.727414$ and is denoted No. 1. The teeth are given half numbers and consequently the tooth between tooth spaces number 1 and 2 is given the number $1\frac{1}{2}$. From eq. (257) we have, as $S_s = 0.727414$ and $\Delta S_r = \Delta S = 0.138555$ radians,

$$S_n = 0.727414 - (n - 1)\, 0.138555$$

The values of S_n are tabulated in the third column in Appendix 13.

The same values of $|S_n|$ and thus also of $|\varphi|$ are valid for a tooth in one quadrant and a tooth space in the other quadrant, although with opposite signs. The table is therefore so arranged that such pairs of values are listed together. Each value of S_n and φ refers not only to a tooth or a tooth space in the single-period L-wheel, but also in the two-period R-wheel. Therefore, if also the tooth space lying at the end of the max. radius for the R-wheel is denoted No. 1, the table includes both wheels.

After having calculated the values of S_n in column 3, the corresponding values of φ have been evaluated by interpolation in the computed table Appendix 11, giving S_n as a function of φ. Generally, higher degree interpolation must be used. In this case, however, linear interpolation will suffice because the second differences are small and in no case greater than 4. Next the values of $\sin\varphi$ and $\cos\varphi$ are found and tabulated in columns 5 and 6 and the values r_1 for the L-wheel and r_2 for the R-wheel have been determined according to the equations of the curves

$$\tan(r_1 - a_1) = \tan(r_2 - \varkappa_2 a_1) = \tan a_1 \sin\alpha_1 \sin\varphi_r$$

These calculations are carried out in columns 7 to 10.

Further, Φ is determined by using the formula in Appendix 5

$$\tan\Phi = \frac{\tan\alpha_1 \cos(r_1 - a_1)\cos\varphi_r}{\cos a_1}$$

These calculations are carried out in columns 11 to 13.

Finally the polar angles $\vartheta_1 = \vartheta_L$ and $\vartheta_2 = \vartheta_R$ are determined in columns 14 to 20 from the equations of the curves

$$\tan\frac{\vartheta_1}{2} = \tan\frac{90° - \alpha_1}{2}\tan\frac{90° - \varphi_r}{2}$$

and

$$\tan\vartheta_2 = \tan\frac{90° - \alpha_2}{2}\tan\frac{90° - \varphi_r}{2}$$

In Appendix 13, all radii, polar angles and angles between the radius vector and the normal for both curves have now been tabulated.

Further, an auxiliary table, Appendix 12, has been prepared giving the values of sin r_1, cos r_1, sin r_2, cos r_2, sin Φ, cos Φ, sin $r_1 \cos \Phi$ and sin $r_2 \cos \Phi$, which are required for the following calculations of the machine settings.

3) Calculation of the machine settings for cutting the tooth spaces and the tips of the teeth and the determination of the required milling cutters

The calculations of the machine settings for cutting the central cuts of the tooth spaces and the tips of the teeth have been carried out in Appendix 14 for the single-period wheel, and in Appendix 15 for the two-period wheel.

Since for both wheels Θ_s as well as $\vartheta_s = 0$, we obtain from eq. (238) or Appendix 2

$$\Theta = \vartheta$$

Further we have from eqs. (260) and (261) or Appendix 10

$$\sin B_{th} = \frac{m}{R_{sph}} = \frac{3}{68} = 0.044118 \text{ or } B_{th} = 2.529°$$

and, if the second equation in (261) is chosen (which seems to be the most attractive),

$$A = \frac{180°}{\pi} \cdot \frac{m}{R_{sph}} = 2.528°$$

The calculations have been carried out with the aid of Appendix 10, i.e. by using eqs. (259) for the tooth spaces and eqs. (284) for the tips of the teeth. From these equations are obtained the machine settings v_{vc}, v_{hc} and v_{dc} for milling the central cuts, and also the settings v_{vt}, v_{ht} and v_{dt} for milling the tips of the teeth. The calculations which comprise all tooth spaces and tooth tips for one half of a period should not need any further explanation.

Next the requisite milling cutters for the different tooth spaces are determined in Appendix 16 with the aid of Appendix 10 or eqs. (283). The values of $\Delta\Phi$ and $\Delta\vartheta$ are evaluated with the aid of Appendix 13 and cos r is taken from Appendix 12. In Appendix 16 are also tabulated the values of h_{pc} determined with the aid of Appendix 8 or eq. (270), as also the values of g_p measured on the milling cutters, the values of c_c and of the out-of-centre settings b_{so} obtained from Appendix 8. The length of the tooth face F has been chosen from the relation $F/R_{sph} = 1/6$. Further, with the aid of Appendix 10, the values of the additional rotation of the rotary milling table Δv_{hc} have been calculated and tabulated. Correction according to eqs. (275) has not been introduced here, as its value would be insignificant. Finally the diameters D_c and D_t of the setting discs have been determined by using Appendix 10 or eqs. (264) and (286), after having first determined the distance f for both wheels, i.e. the distance from the centre of the dividing head to the apex of the pitch cone. This distance cannot be determined before the arbor is mounted in the dividing head. It is composed of the measured distance from the centre of the dividing head to the locating face of the arbor and the distance from the apex of the pitch cone to the outside of the boss as taken from the drawing (see Fig. 18).

We obtain the following values:

For the single-period wheel: $f = 118.12$ (measured) $+ 70.00$ (from the drawing) $=$
$= 188.12$ mm

For the two-period wheel: $f = 107.62$ (measured) $+ 58.00$ (from the drawing) $=$
$= 165.62$ mm

It is, however, not necessary to manufacture separate setting discs for the cutting of the tips of the teeth, but the setting discs intended for the tooth spaces may be used. For cutting the tips of the teeth the same cutting table may be used without any alteration, if, after the contour cutter has been made to touch the setting disc with the diameter D_c, the milling-machine table is lowered a distance $\frac{D_t - D_c}{2} = 4.40$ mm for the single-period wheel, and elevated a distance $\frac{D_c - D_t}{2} = 3.54$ mm for the two-period wheel, before the setting scale is adjusted to zero and locked into position.

Further, the settings v_{hcL}, v_{hcR}, l_{cL} and l_{cR} (i.e. the settings when cutting the side cuts) and the settings l_c (for the central cuts) and h_c (which are valid for both central and side cuts) have been evaluated with the aid of Appendix 10 or eqs. (265), (272) and (276) and are listed in Appendix 18, which is valid for both wheels.

Finally, the values l_t and h_t for the height- and cross-setting of the milling-machine table when cutting the tips of the teeth have been calculated in Appendix 17 for both wheels according to Appendix 10 or eqs. (285).

4) Preparation of cutting schedules for cutting the tooth spaces and the tips of the teeth

From the data thus calculated, the cutting schedules intended for the milling-machine operator can be prepared. They are found in the following appendices:

Appendix 19 for the cutting of the tooth spaces of the single-period wheel
" 20 " " " " " " " " " two- " "
" 21 " " " " " contour for both wheels

Contrary to the numbering in Appendices 12—18, where the teeth are given half numbers, tooth spaces as well as teeth are given in the above appendices whole numbers in such a way that tooth No. 1 in Appendix 21 corresponds to No. $1\frac{1}{2}$ in the foregoing appendices. For the first eleven tooth spaces and tips of the teeth corresponding to half a period, the values of the machine settings in the cutting schedules can be taken directly from the foregoing tables, Appendices 14, 15, 17 and 18. The angles v_d and v_h are, however, expressed in revolutions + degrees on the index plates, assuming the ratios of the dividing head and the rotary milling table to be 40/1 and 60/1, respectively. If v_d is divided by 9, the integral quotient then gives the number of revolutions. Multiplying the remainder by 40 gives the degrees. In a similar manner, v_v is divided by 6 and the remainder multiplied by 60. Further, 360° (= 60 revolutions of the index handle) is added to v_h in order to avoid negative values of the settings. In the last column of the schedules for the cutting of the tooth spaces the milling cutter numbers taken from Appendix 16 have also been entered.

After having completed the cutting tables for half a period of each wheel, the remaining values for the other parts of the wheels may easily be obtained by reasons of symmetry. If m is an arbitrary integer, then we obtain the relations below between tooth space No. $(12+m)$ or tooth tip No. $(11+m)$ and a corresponding tooth space or tooth tip No. $(11-m)$.

For both the single-period and the two-period wheel we have

	for tooth space No. $(12+m)$ or tooth tip No. $(11+m)$		for tooth space or tooth tip No. $(11-m)$
v_d	$= \frac{360^\circ\ (= 40 \text{ turns on the index handle})}{p} - v_d$		
v_h for the central cut	$= 720^\circ\ (= 120$ turns on the index handle)	$- v_h$ for the central cut	
v_h for the left-hand cut	$= 720^\circ$ do.	$- v_h$ for the right-hand cut	
v_h for the right-hand cut	$= 720^\circ$ do.	$- v_h$ for the left-hand cut	
v_h for the tip of the tooth	$= 720^\circ$ do.	$- v_h$ for the tip of the tooth	
v_v	$= v_v$		
h	$= h$		
l for the central cut	$= -l$ for the central cut		
l for the left-hand cut	$= -l$ for the right-hand cut		
l for the right-hand cut	$= -l$ for the left-hand cut		
l for the tip of the tooth	$= -l$ for the tip of the tooth		
Z_c	$= Z_c$		

For the two-period wheel we have in addition

	for tooth space or tooth tip No. $(21+m)$		for tooth space or tooth tip No. m
v_d	$= \frac{360^\circ\ (= 40 \text{ turns on the index handle})}{p} + v_d$		
v_h	$= v_h$		
v_v	$= v_v$		
h	$= h$		
l	$= l$		
Z_c	$= Z_c$		

By using the above laws of symmetry, the rest of the cutting schedules Appendices 19, 20, and 21 can easily be prepared, and by using these cutting schedules no difficulty should be encountered in cutting the contour and the tooth spaces of the wheels.

Previously it has been assumed that the different milling cutters used have exactly the same outer diameter and the same thickness of the boss, and also that the centre line of the flanks coincides with the centre line of the cutter. If this should not be the case, the machine settings will have to be modified accordingly.

A non-circular bevel gear train comprising two single-period wheels and one two-period wheel has been manufactured according to the above calculations, and is shown on Figs. 27a and b. The shaft angle between the single-period and the two-period wheel is according to the premises equal to 90°. As $a_1 = 29.670^\circ$, the shaft angle between the two single-period wheels becomes 59.340°. Fig. 28 shows the tooth cutting of the single-period wheel and Fig. 14 of the two-period wheel.

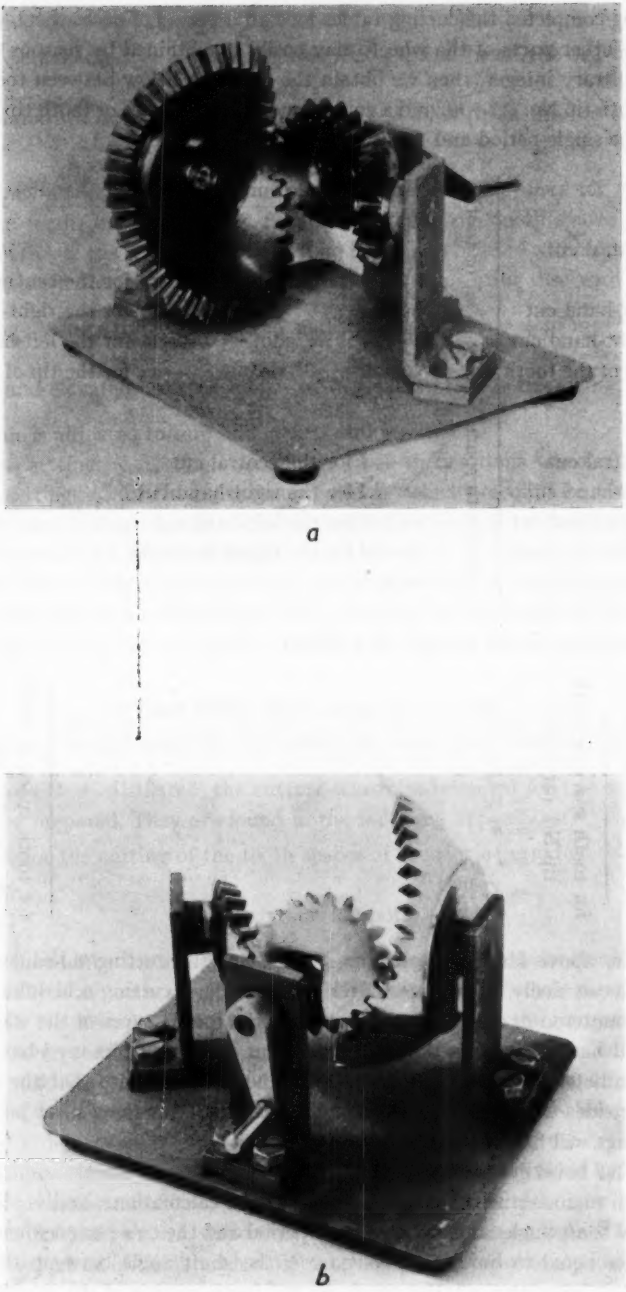


Fig. 27. Non-circular bevel gear train comprising two single-period, and one two-period, wheels.

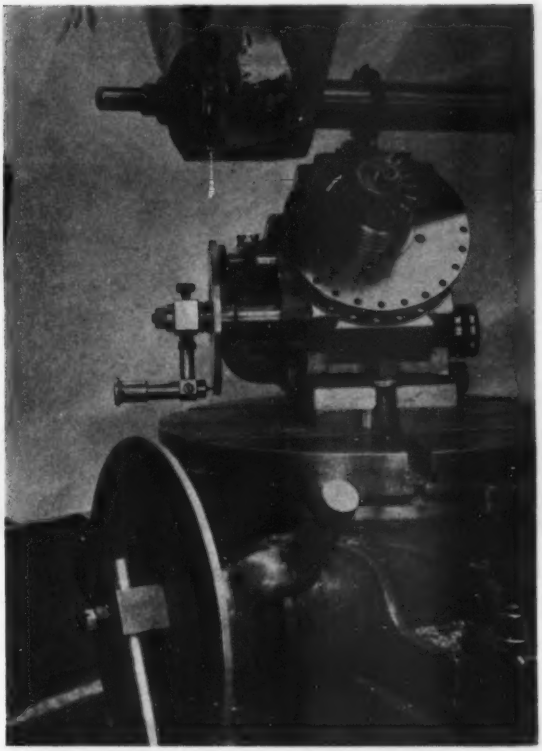


Fig. 28. Tooth-cutting of the single-period non-circular bevel gear wheel shown in Fig. 27.

b) A twin bevel gear

Fig. 29 shows a bevel gear, the ratio curve of which is calculated in Chapter VI:D, pp. 116—118, and illustrated in Fig. 11.

The wheels have the following data:

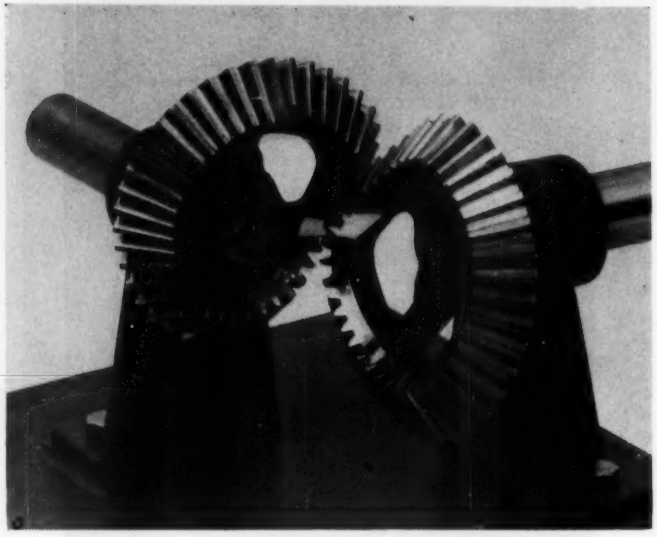
$$R_{sph} = 82.75 \text{ mm}$$
$$m = 3$$
$$\frac{F}{R_{sph}} = \frac{24.75}{82.75} \approx 0.3$$

The part of the pitch curve formed by the larger spherical circle comprises 43.707° for one half of the wheel and has

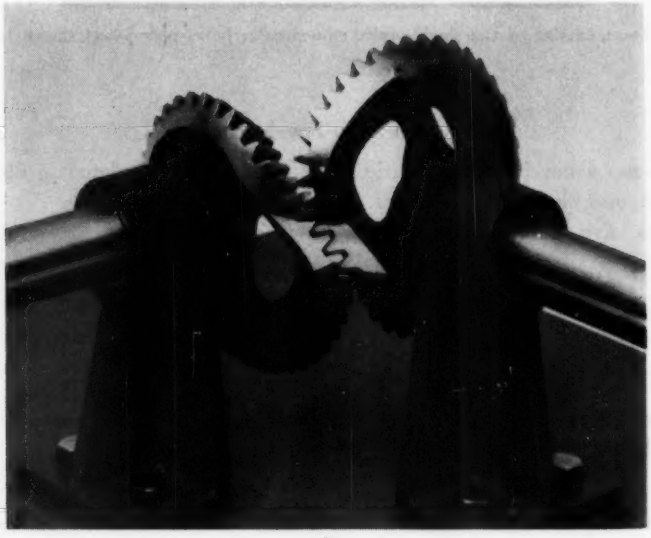
$$R = 56.100°$$

The part of the pitch curve formed by the smaller spherical circle comprises 65.041° for one half of the wheel and has

$$R = 33.900°$$



a



b

Fig. 29. A twin bevel gear the ratio curve of which is shown in Fig. 11.

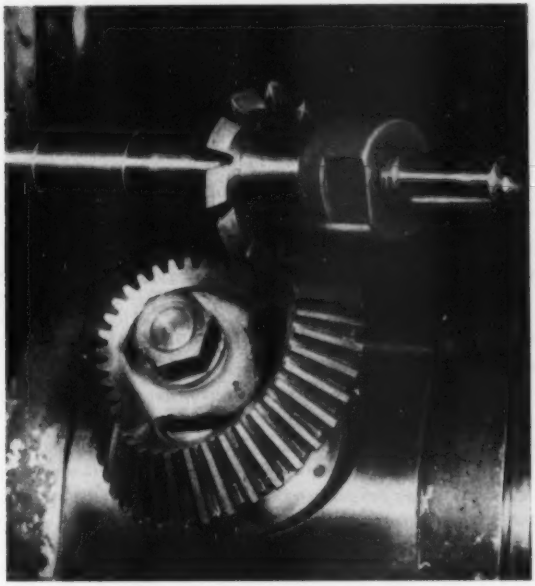


Fig. 30. Tooth cutting of the bevel gear wheels shown in Fig. 29.

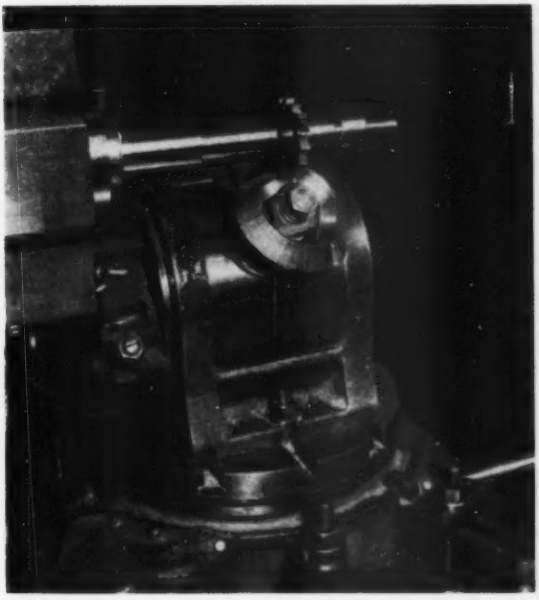


Fig. 31. Cutting the tips of the teeth of the bevel gear wheels shown in Fig. 29.

The part of the pitch curve formed by the rolling curves to a spherical ellipse comprises 71.252° for each part and has

$$a_1 = 19.333^\circ$$
$$\alpha_1 = 34.000^\circ$$
$$a_v = 45.000^\circ$$
$$\alpha_v = 11.314^\circ$$
$$\nu = 2.52624$$

Fig. 30 shows the tooth cutting and Fig. 31 the cutting of the tips of the teeth.

Finally, it should be mentioned that most of the calculations could with advantage be carried out in an electronic digital computer. The table, for example, giving the arc length as a function of φ for the last-treated gear (corresponding to Appendix 12) has been calculated in such a computer.

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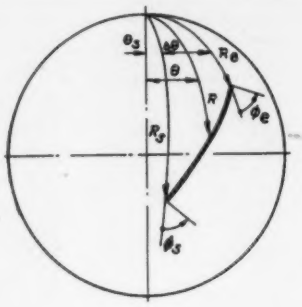
Scheme showing the interrelationship of spherical conic section rolling curves

Numbers in parenthesis indicate the equation number of the corresponding curve. The upper number refers to the parametric equation, and the lower number to the non-parametric equation.

- **Spherical conic section rolling curves**
  - **a_1 real and e real** — Rolling curves to the spherical conic sections with real axes
    - $e \neq 0$
      - $e \neq 0$ and real
        - $e > 1$ — Rolling curves to the imaginary spherical ellipse, v imaginary
          - $\alpha_1 \neq 90°$
            - $a_v \neq 0$ or $90°$ — (144) / (146)
            - $a_v = 90°$ (Crown-wheel curve, spherical sinh-curve) — (148) / (150)
            - $a_v = 0$ (Spherical sinh-curve) — (153) / (155)
          - $\alpha_1 = 90°$ — a is here indeterminate (spherical sinh-curve having Φ = constant) — (158) / (160) or (162)
        - $e \leq 1$ — Rolling curves to the real spherical conic sections
          - v real
            - $a_v \neq 90°$ — (107) / (109)
            - $a_v = 90°$ (Crown-wheel curve, spherical cos-curve)* — (120) / (122)
          - v imaginary
            - $a_v \neq 0$ — (115) / (117)
            - $a_v = 0$ (Spherical cosh-curve) — (125) / (127)
          - $v = 0$ — (130) / (132)
    - $e = 0$ — Spherical circles
      - v is here indeterminate
        - $a_v \neq 90°$ (Spherical circle) — {(135)
        - $a_v = 90°$ (Crown-wheel curve, great circle) — {(135)
  - **a_1 imaginary** — Rolling curves to the spherical conic sections with imaginary axes
    - $e \neq 0$ and imaginary — Rolling curves to the imaginary spherical hyperbola
      - v real
        - $\bar{a}_v \neq 90°$ — (171) / (174)
        - $\bar{a}_v = 90°$ (Crown-wheel curve, spherical cos-curve) — (182) / (184)
      - v imaginary
        - $\bar{a}_v \neq 0$ — (177) / (179)
        - $\bar{a}_v = 0$ (Spherical cosh-curve) — (187) / (189)
      - $v = 0$ — (192) / (194)
    - $e = 0$ (v is imaginary) — Rolling curves to the imaginary spherical conic sections when $e \to 0$
      - $|v| \neq 0$
        - $a_v \neq 90°$ — (205) / (207)
        - $a_v = 90°$ (Crown-wheel curve, spherical exponential curve) — (209) / (211)
      - $|v| = 0$
        - $a_v = 0$ (Spherical helix) — (213) / (214)

* If $a_1 = 90°$ and $\alpha_1 = 0$, then the crown-wheel curve is an eccentric great circle

The given curve is divided into a convenient number of monotonic arcs, i. e. arcs of such a nature that Φ does not cha



The given curve is expressed by

$$\tan R = R(\Theta)$$

where $0 < R < 180°$.

The substitute curve is written

$$\tan r = r(\vartheta)$$

where $-180° < r < +180°$.

$$\tan \Phi = -\frac{1}{\sin R}\cdot\frac{dR}{d\Theta} = -\frac{1}{\sin r}\cdot\frac{dr}{d\Theta}$$

$$\tan \overline{\Phi} = \frac{\tan \Phi}{\cos R} = -\frac{1}{\tan R}\cdot\frac{d(\tan R)}{d\Theta} = \frac{\tan \Phi}{\cos r} =$$

$$= -\frac{1}{\tan r}\cdot\frac{d(\tan r)}{d\vartheta} = -\frac{1}{\cos|C_{L-R}| - u}\cdot\frac{du}{d\Theta} =$$

$$= \frac{1}{\left(\frac{1}{u}\right)^2 \cos|C_{L-R}| - \frac{1}{u}}\cdot\frac{d\left(\frac{1}{u}\right)}{d\Theta}$$

For external meshing u is negative.

$$\Delta\Theta = \Theta_e - \Theta_s$$

$$\vartheta = \Theta - \Theta_s + \vartheta_s$$

Whether the substitute curve corresponds to the main curve or the antipode curve can be determined by using the rules in Appendix 3.

---

[1]) The sequence of the signs in the equations of the substitute curves given in the last column corresponds to the sequence given in the column marked [1]). Thus, the upper expression and sign corresponds to the greater branch or the tan-branch, and the lower expression and sign to the smaller branch or the cot-branch.

[2]) Upper sign if $k_4 = -1$. Lower sign if $k_4 = +1$. ϑ_s in radians.

[3]) For the signs in the equations the following is valid:
Upper sign in $\mp |\nu|\vartheta$ if $k_4 = -1$ giving an A-branch.
Lower sign in $\mp |\nu|\vartheta$ if $k_4 = +1$ giving a B-branch.
Upper sign in $\mp e$ if $\tan \overline{\Phi}_s > \tan \overline{\Phi}_e$ giving a tan-A or cot-B branch.
Lower sign in $\mp e$ if $\tan \overline{\Phi}_s < \tan \overline{\Phi}_e$ giving a tan-B or cot-A branch.

For curves of this type we also have

$$|\nu| = \left|\frac{\tan R_s \tan \overline{\Phi}_e - \tan R_e \tan \Phi_s}{\tan R_s - \tan R_e}\right| =$$

$$= \left|\frac{1}{\Delta\Theta}\ln\frac{\tan R_s \tan \Phi_e}{\tan R_e \tan \Phi_s}\right|$$

[4]) Upper sign for $|\nu|$ if $\tan \Phi_s < 0$ giving an A-branch.
Lower sign for $|\nu|$ if $\tan \Phi_s > 0$ giving a B-branch.
$\tan R_s < 0$ gives a tan-A or cot-B branch.
$\tan R_s > 0$ gives a cot-A or tan-B branch.

[5]) For curves of this type we also have

$$\Delta\Theta = \frac{2(\tan R_s - \tan R_e)}{\tan R_s \tan \overline{\Phi}_e + \tan R_e \tan \Phi_s}$$

[6]) Also $k_2 = \dfrac{2\Delta\Theta \tan R_e}{\tan R_s \tan \overline{\Phi}_e - \tan R_e \tan \Phi_s} - \vartheta_s^2$

[7]) Corresponding to tan-branch A.

Determination of ν or $\lvert\nu\rvert$			Determination of ϑ_s			Determinati
Evaluate	If	obtain ν or $\lvert\nu\rvert$ from	Evaluate	If	obtain ϑ_s from	
$k_3 = \dfrac{\Delta\Theta}{2}\cdot\dfrac{\tan R_s \tan \Phi_e + \tan R_e \tan \Phi_s}{\tan R_s - \tan R_e}$ (k_3 will always be positive)	$k_3 < \dfrac{180°}{\pi} = 1$ radian	$\tan\left(\nu\dfrac{\Delta\Theta}{2}\right) = \dfrac{1}{k_3}\left(\nu\dfrac{\Delta\Theta}{2}\right)$ Must be solved by inspection. See curve in Appendix 2 in NCCG (Ref. 12).	$k_4 = \dfrac{2\nu\tan R_e(\tan R_s - \tan R_e)\tan \Phi_s}{\tan^2 R_s \tan^2 \Phi_e - \tan^2 R_e \tan^2 \Phi_s + \nu^2(\tan R_s - \tan R_e)^2}$	—	$\tan \nu\vartheta_s = k_4$ The solution is chosen in the quadrant which makes k_2 positive	$k_2 = \dfrac{\tan R_s \cos \nu\vartheta_s - \tan R_e \cos \nu(\vartheta_s + \Delta\Theta)}{\tan R_s - \tan R_e}$
	$k_3 > \dfrac{180°}{\pi} = 1$ radian	$\tanh\left(\lvert\nu\rvert\dfrac{\Delta\Theta}{2}\right) = \dfrac{1}{k_3}\left(\lvert\nu\rvert\dfrac{\Delta\Theta}{2}\right)$ Must be solved by inspection. See curve in Appendix 2 in NCCG (Ref. 12).	$k_4 = \dfrac{2\lvert\nu\rvert\tan R_e(\tan R_s - \tan R_e)\tan \Phi_s}{\tan^2 R_s \tan^2 \Phi_e - \tan^2 R_e \tan^2 \Phi_s - \lvert\nu\rvert^2(\tan R_s - \tan R_e)^2}$	$-1 < k_4 < +1$	$\tanh\lvert\nu\rvert\vartheta_s = k_4$	$k_2 = \dfrac{\tan R_s \dfrac{\cosh\lvert\nu\rvert\vartheta_s}{\sinh} - \tan R_e \dfrac{\cosh\lvert\nu\rvert(\vartheta_s + \Delta\Theta)}{\sinh}}{\tan R_s - \tan R_e}$
				$\lvert k_4\rvert > 1$	$\coth\lvert\nu\rvert\vartheta_s = k_4$	
				$k_4 = \mp 1$ $\tan \overline{\Phi}_s = \tan \Phi_e$	$\vartheta_s = \pm\dfrac{1}{\lvert\nu\rvert}\ln\left\lvert\dfrac{\tan R_s}{\tan R_s - \tan R_e}\left(1 -\right.\right.$	
					$\vartheta_s = 0$	—
$k_3 = \dfrac{180°}{\pi} = 1$ radian	$\nu = \lvert\nu\rvert = 0$	—	$\tan R_s \tan \Phi_e + \tan R_e \tan \Phi_s$ [5])		$\vartheta_s = \dfrac{\Delta\Theta \tan R_e \tan \Phi_s}{\tan R_s \tan \Phi_e - \tan R_e \tan \Phi_s}$	$k_2 = \dfrac{2\vartheta_s - \vartheta_s^2}{}$
				$\dfrac{\tan R_s}{\tan \Phi_e} = \tan R_e \tan \Phi_s$	$\vartheta_s = \dfrac{1}{\tan \Phi_s}$	—
k_2 is indeterminate	ν is indeterminate	—	—	—	—	—

Replacing a given spherical curve by arcs of spherical conic section ro

t change its sign. For each arc $\tan R_s$, $\tan R_e$, $\tan \Phi_s$, $\tan \Phi_e$ and $\triangle\Theta$ are determined according to the figure and $\tan \bar{\Phi}_s$ and $\tan \bar{\Phi}_e$ e

ination of k_2	Determination of k_1			Equation of the substitute curve $\tan \nu =$	If	$\left(0 \leq \frac{a_\nu}{\bar{a}_\nu} \leq 9\right.$	
	If [1]	take k_1	$k_1 =$				
$k_2 = \frac{\tan R_s \cos \nu \Theta_s - \tan R_e \cos \nu (\Theta_s + \triangle\Theta)}{\tan R_s - \tan R_e}$	$k_2 \geqq 0$	pos.	$\frac{\tan R_s \tan R_e}{\tan R_s - \tan R_e} [\cos \nu \Theta_s - \cos \nu (\Theta_s + \triangle\Theta)]$	$\frac{k_1}{k_2 - \cos \nu \Theta}$	$k_2 \neq 0$	$\left.\begin{matrix}\tan 2 a_\nu \\ \text{or} \\ \tan 2 \bar{a}_\nu\end{matrix}\right\} = \frac{2 k_1 k_2}{k_2^2 - k_1^2 - 1}$	$\sin \alpha_\nu = \tan \frac{\alpha_T}{2}$ where $\sin \alpha_T = \frac{2 k_2}{k_2^2 + k_1^2 + 1}$
					$k_2 = 0$	$\left.\begin{matrix}a_\nu \\ \text{or} \\ \bar{a}_\nu\end{matrix}\right\} = 90^\circ$	$\alpha_\nu = 0$
$\tan R_s - \tan R_e$ cosh if $-1 < k_2 < +1$	$k_2 \geqq 0$ $k_2 < 0$	neg. pos.	$\frac{\tan R_s \tan R_e}{\tan R_s - \tan R_e} \left[\frac{\cosh \lvert\nu\rvert}{\sinh \lvert\nu\rvert} \Theta_s - \frac{\cosh \lvert\nu\rvert}{\sinh \lvert\nu\rvert} (\Theta_s + \triangle\Theta)\right]$	$\frac{k_1}{k_2 - \cosh \lvert\nu\rvert \Theta}$ cosh if $-1 < k_2 < +1$	$k_2 \neq 0$	$\left.\begin{matrix}\tan 2 a_\nu \\ \text{or} \\ \tan 2 \bar{a}_\nu\end{matrix}\right\} = \frac{2 \lvert k_1\rvert \lvert k_2\rvert}{k_2^2 - k_1^2 + 1}$	$\sin \alpha_\nu = \tan \frac{\alpha_T}{2}$ where $\sin \alpha_T = \frac{2 \lvert k_2\rvert}{k_2^2 + k_1^2 + 1}$
					$k_2 = 0$	$\left.\begin{matrix}a_\nu \\ \text{or} \\ \bar{a}_\nu\end{matrix}\right\} = 0$	$\alpha_\nu = 0$
sinh if $\lvert k_2\rvert > 1$	$k_2 \leqq 0$ $k_2 > 0$	neg. pos.		$\frac{k_1}{k_2 - \sinh \lvert\nu\rvert \Theta}$ sinh if $\lvert k_2\rvert > 1$	$k_2 \neq 0$	$\tan 2 a_\nu = \frac{2 k_1 k_2}{k_2^2 - k_1^2 + 1}$	$\tan 2 \alpha_\nu = \frac{2 \lvert k_2\rvert}{k_1^2 + k_2^2 - 1}$
					$k_2 = 0$ $\lvert k_1\rvert > 1$	$a_\nu = 90^\circ$	$\alpha_\nu = 0$
					$k_2 = 0$ $\lvert k_1\rvert < 1$	$a_\nu = 0$	$\alpha_\nu = 90^\circ$
					$k_2 = 0$ $\lvert k_1\rvert = 1$	a_ν may be chosen arbitrarily	$\alpha_\nu = 90^\circ - a_\nu$ Choose $a_\nu = R_s$ or $a_\nu = 180^\circ - R_s$
$\frac{1}{e}\left(1 - \frac{\tan \Phi_e}{\tan \Phi_s}\right)$ [2]	—	pos.	$\pm \frac{\tan R_e \tan \Phi_s - \tan R_s \tan \Phi_e}{\tan \Phi_s - \tan \Phi_e}$	$\frac{k_1}{1 \mp e^{\mp \lvert\nu\rvert \Theta}}$ [3]	—	$\tan a_\nu = k_1$	$\tan \alpha_\nu = \frac{k_1}{\tanh \lvert a_1\rvert}$ (to be determined after $\lvert a_1\rvert$)
—	—	pos.	$\tan R_s$	$k_1 e^{\pm \lvert\nu\rvert \Theta}$ [4]	—	$a_\nu = 90^\circ$	$\alpha_\nu = 90^\circ$
$k_2 = \frac{2 \Theta_s}{\tan \bar{\Phi}_s} - \Theta_s^2$ [5]	—	pos.	$\pm \frac{2 \Theta_s \tan R_s}{\tan \Phi_s} = \pm \frac{2 \triangle\Theta \tan R_s \tan R_e}{\tan R_s \tan \bar{\Phi}_e - \tan R_e \tan \Phi_s}$	$\frac{k_1}{k_2 + \Theta^2}$	—	$\left.\begin{matrix}\tan 2 a_0 \\ \text{or} \\ \tan 2 \bar{a}_0\end{matrix}\right\} = \frac{k_1}{k_2}$	$\alpha_0 = 90^\circ$
—	—	pos.	$\left\lvert\frac{\tan R_s}{\tan \Phi_s}\right\rvert$	$\frac{k_1}{\Theta}$ [7]	—	$a_0 = 0$	$\alpha_0 = 0$
—	—	pos.	$\tan R_s$	$k_1 = \tan R_s$ or $\nu = R_s$	—	$a_\nu = R_s$	$\sin a_\nu = \nu$

f spherical conic section rolling curves

ed according to the figure and $\tan \bar{\Phi}_s$ and $\tan \bar{\Phi}_e$ evaluated from the formulae below. The correspo

Evaluate a_ν or $\bar{a}_\nu$, α_ν, a_1 or $|a_1|$, α_1 and $\varkappa_\nu$ or $|$

$\left(0 \leq \frac{a_\nu}{\bar{a}_\nu} \leq 90^\circ,\ 0 \leq \alpha_T \leq 90^\circ,\ 0 \leq \alpha_\nu \leq 90^\circ,\ 0 \leq a_1 \leq 90^\circ,\ 0 \right.$

$\left.\begin{matrix}\tan 2\,a_\nu \\ \text{or} \\ \tan 2\,\bar{a}_\nu\end{matrix}\right\} = \frac{2\,k_1\,k_2}{k_2^2 - k_1^2 - 1}$	$\sin \alpha_\nu = \tan \frac{\alpha_T}{2}$ where $\sin \alpha_T = \frac{2\,k_2}{k_2^2 + k_1^2 + 1}$	If $\nu \geq \sin a_\nu \cos \alpha_\nu$	co $= \cos \alpha_\nu \sqrt{\frac{\cos^2 a_\nu}{\cos^2 a_\nu +}}$
		If $\nu < \sin \bar{a}_\nu \cos \alpha_\nu$	$\cos \alpha_1 = \frac{1}{\cos \alpha_\nu} \sqrt{\frac{\cos^2 \bar{a}_\nu +}{\cos^2 \bar{a}_\nu}}$
$\left.\begin{matrix}a_\nu \\ \text{or} \\ \bar{a}_\nu\end{matrix}\right\} = 90^\circ$	$\alpha_\nu = 0$	If $\nu \geq 1$	$\cos \alpha_1 = \sqrt{\frac{1 + k}{\nu^2 + k}}$
		If $\nu < 1$	$\cos \alpha_1 = \sqrt{\frac{\nu^2 + k}{1 + k}}$
$\left.\begin{matrix}\tan 2\,a_\nu \\ \text{or} \\ \tan 2\,\bar{a}_\nu\end{matrix}\right\} = \frac{2\,\lvert k_1\rvert\,\lvert k_2\rvert}{k_1^2 - k_2^2 + 1}$	$\sin \alpha_\nu = \tan \frac{\alpha_T}{2}$ where $\sin \alpha_T = \frac{2\,\lvert k_2\rvert}{k_2^2 + k_1^2 + 1}$	If $\lvert\nu\rvert \geq \frac{\sin a_\nu}{\tan \alpha_\nu}$	$\cos \alpha_1 = \frac{1}{\tan \alpha_\nu} \sqrt{\frac{1 - \cos^2}{\lvert\nu\rvert^2 + \cos}}$
		If $\lvert\nu\rvert < \frac{\sin \bar{a}_\nu}{\tan \alpha_\nu}$	$\cos \alpha_1 = \tan \alpha_\nu \sqrt{\frac{\lvert\nu\rvert^2 + \cos}{1 - \cos^2}}$
$\left.\begin{matrix}a_\nu \\ \text{or} \\ \bar{a}_\nu\end{matrix}\right\} = 0$	$\alpha_\nu = 0$	If $\lvert\nu\rvert \geq \lvert k_1\rvert$	$\tan \alpha_1 = \sqrt{\frac{\lvert\nu\rvert^2 -}{1 + k}}$
		If $\lvert\nu\rvert < \lvert k_1\rvert$	$\sin \alpha_1 = \sqrt{\frac{k_1^2 - \lvert}{1 + k}}$
$\tan 2\,a_\nu = \frac{2\,k_1\,k_2}{k_2^2 - k_1^2 + 1}$	$\tan 2\,\alpha_\nu = \frac{2\,\lvert k_2\rvert}{k_1^2 + k_2^2 - 1}$	—	$\cos \alpha_1 = \frac{1}{\cos \alpha_\nu} \sqrt{\frac{\cos^2 a_\nu -}{\cos^2 a_\nu +}}$
$a_\nu = 90^\circ$	$\alpha_\nu = 0$	—	$\cos \alpha_1 = \sqrt{\frac{k_1^2 -}{k_1^2 + \lvert\nu}}$
$a_\nu = 0$	$\alpha_\nu = 90^\circ$	—	$\tan \alpha_1 = \sqrt{\frac{\lvert\nu\rvert^2 +}{1 - k}}$
a_ν may be chosen arbitrarily	$\alpha_\nu = 90^\circ - a_\nu$ Choose $a_\nu = R_s$ or $a_\nu = 180^\circ - R_s$	—	$\alpha_1 = 90^\circ$ Also $\lvert\nu\rvert = \lvert\tan \Phi\rvert = \lvert\cos R \tan \bar{\Phi}$
$\tan a_\nu = k_1$	$\tan \alpha_\nu = \frac{k_1}{\tanh \lvert a_1\rvert}$ (to be determined after $\lvert a_1\rvert$)	—	$\alpha_1 = 0$
$a_\nu = 90^\circ$	$\alpha_\nu = 90^\circ$	—	$\alpha_1 = 0$ Also $\lvert\nu\rvert = \lvert\tan \bar{\Phi}\rvert = \text{cons}$
$\left.\begin{matrix}\tan 2\,a_0 \\ \text{or} \\ \tan 2\,\bar{a}_0\end{matrix}\right\} = \frac{k_1}{k_2}$	$\alpha_0 = 90^\circ$	If $\tan a_0 \leq \frac{2}{k_1}$ then also $\frac{k_1^2}{4} - k_2 \leq 1$	$\sin \alpha_1 = \tan a_0 \sqrt{\frac{2}{\tan 2\,\tan}}$
		If $\tan \bar{a}_0 > \frac{2}{k_1}$ then also $\frac{k_1^2}{4} - k_2 > 1$	$\tan \alpha_1 = \tan \bar{a}_0 \sqrt{\frac{k_1 -}{2\,\tan}}$
$a_0 = 0$	$\alpha_0 = 0$	—	$\alpha_1 = 0$
$a_\nu = R_s$	$\sin a_\nu = \nu$	—	$\alpha_1 = 0$

orresponding arcs of spherical conic section rolling curves are then derived according to the table.

$\varkappa_\nu$ or $\lvert\varkappa_\nu\rvert$ from 90°, $0 \le \lvert a_1\rvert$, $0 \le a_1 \le 90^\circ$)			Substitute curve is a rolling curve to	Eqs. No.
$\sqrt{\cos^2 a_\nu + \tan^2 \alpha_\nu}$ / $\sqrt{\cos^2 a_\nu + \nu^2 \tan^2 \alpha_\nu}$	$\sin a_1 = \dfrac{\sin a_\nu \cos \alpha_\nu}{\nu \cos \alpha_1}$	$\varkappa_\nu = \dfrac{a_\nu}{a_1}$	a real spherical conic section ν real	(107) (109) (108)
$\sqrt{\cos^2 \bar{a}_\nu + \nu^2 \tan^2 \alpha_\nu}$ / $\sqrt{\cos^2 \bar{a}_\nu + \tan^2 \alpha_\nu}$	$\cosh \lvert a_1\rvert = \dfrac{\sin \bar{a}_\nu \cos \alpha_1 \cos \alpha_\nu}{\nu}$	$\lvert\varkappa_\nu\rvert = \dfrac{90^\circ - \bar{a}_\nu}{\lvert a_1\rvert}$	an imaginary spherical hyperbola ν real	(171) (174) (172)
$\sqrt{1 + k_1^2}$ / $\sqrt{\nu^2 + k_1^2}$	$\sin a_1 = \dfrac{1}{\nu \cos \alpha_1}$	$\varkappa_\nu = \dfrac{90^\circ}{a_1}$ If $\nu = 1$, then $\tan a_1 \sin \alpha_1 = \dfrac{1}{k_1}$	a real spherical conic section ν real (Crown-wheel curve)	(120) (122) (121)
$\sqrt{\nu^2 + k_1^2}$ / $\sqrt{1 + k_1^2}$	$\cosh \lvert a_1\rvert = \dfrac{\cos \alpha_1}{\nu}$	$\varkappa_\nu = 0$	an imaginary spherical hyperbola ν real (Crown-wheel curve)	(182) (184) (183)
$\sqrt{\quad - \cos^2 a_\nu \cos^2 \alpha_\nu}$ / $\sqrt{\lvert\nu\rvert^2 + \cos^2 a_\nu \cos^2 \alpha_\nu}$	$\sin a_1 = \dfrac{\sin a_\nu}{\lvert\nu\rvert \cos \alpha_1 \tan \alpha_\nu}$	$\varkappa_\nu = \dfrac{a_\nu}{a_1}$	[1]) a real spherical conic section ν imaginary	(115) (117) (116)
$\sqrt{\lvert\nu\rvert^2 + \cos^2 \bar{a}_\nu \cos^2 \alpha_\nu}$ / $\sqrt{\quad - \cos^2 \bar{a}_\nu \cos^2 \alpha_\nu}$	$\cosh \lvert a_1\rvert = \dfrac{\sin \bar{a}_\nu \cos \alpha_1}{\lvert\nu\rvert \tan \alpha_\nu}$	$\lvert\varkappa_\nu\rvert = \dfrac{90^\circ - \bar{a}_\nu}{\lvert a_1\rvert}$	[1]) an imaginary spherical hyperbola ν imaginary	(177) (179) (178)
$\sqrt{\lvert\nu_1\rvert^2 - k_1^2}$ / $\sqrt{1 + k_1^2}$	$\cos a_1 = \dfrac{\tan \alpha_1}{\lvert\nu\rvert}$	$\varkappa_\nu = 0$ If $\lvert\nu\rvert = 0$, then $\tan a_1 \sin \alpha_1 = \lvert k_1\rvert$	[1]) a real spherical conic section ν imaginary (The rolling curve having $a_\nu = 0$)	(125) (127) (126)
$\sqrt{k_1^2 - \lvert\nu\rvert^2}$ / $\sqrt{1 + k_1^2}$	$\sinh \lvert a_1\rvert = \dfrac{\sin \alpha_1}{\lvert\nu\rvert}$	$\lvert\varkappa_\nu\rvert = \dfrac{90^\circ}{\lvert a_1\rvert}$	[1]) an imaginary spherical hyperbola — ν imaginary (The rolling curve having $\bar{a}_\nu = 0$)	(187) (189) (188)
$\sqrt{\cos^2 a_\nu - \sin^2 \alpha_\nu}$ / $\sqrt{\cos^2 a_\nu + \lvert\nu\rvert^2 \sin^2 \alpha_\nu}$	$\sinh \lvert a_1\rvert = \dfrac{\sin a_\nu}{\lvert\nu\rvert \cos \alpha_1 \cos \alpha_\nu}$	$\lvert\varkappa_\nu\rvert = \dfrac{a_\nu}{\lvert a_1\rvert}$	[1]) an imaginary spherical ellipse	(144) (146) (145)
$\sqrt{k_1^2 - 1}$ / $\sqrt{k_1^2 + \lvert\nu\rvert^2}$	$\sinh \lvert a_1\rvert = \dfrac{1}{\lvert\nu\rvert \cos \alpha_1}$	$\lvert\varkappa_\nu\rvert = \dfrac{90^\circ}{\lvert a_1\rvert}$	[1]) an imaginary spherical ellipse (Crown-wheel curve)	(148) (150) (149)
$\sqrt{\lvert\nu\rvert^2 + k_1^2}$ / $\sqrt{1 - k_1^2}$	$\cosh \lvert a_1\rvert = \dfrac{\tan \alpha_1}{\lvert\nu\rvert}$	$\lvert\varkappa_\nu\rvert = 0$	[1]) an imaginary spherical ellipse (The rolling curve having $a_\nu = 0$)	(153) (155) (154)
90° 0 $\Re \tan \bar{\Phi}\rvert = \text{constant}$	$\lvert a_1\rvert = \infty$	$\lvert\varkappa_\nu\rvert = 0$	[1]) an imaginary spherical ellipse having $\alpha_1 = 90^\circ$	(158) (160) or (162) (159)
$= 0$	$\sinh \lvert a_1\rvert = \dfrac{\sin a_\nu}{\lvert\nu\rvert}$	$\lvert\varkappa_\nu\rvert = \dfrac{a_\nu}{\lvert a_1\rvert}$	[3]) an imaginary spherical conic section having $e = 0$	(205) (207) (206)
$= 0$ ∞ $\rvert = \text{constant}$	$\sinh \lvert a_1\rvert = \dfrac{1}{\lvert\nu\rvert}$	$\lvert\varkappa_\nu\rvert = \dfrac{90^\circ}{\lvert a_1\rvert}$	[4]) an imaginary spherical conic section having $e = 0$ (Crown-wheel curve; spherical exponential curve)	(209) (211) (210)
$\sqrt{\dfrac{2}{\tan a_0} - k_1}$ / $2 \tan a_0 + k_1$	$\tan a_1 = \dfrac{\tan a_0}{\sin \alpha_1}$ If $\tan a_0 = \dfrac{2}{k_1}$, then $\tan a_1 \sin \alpha_1 = \dfrac{2}{k_1}$	$\varkappa_0 = \dfrac{a_0}{a_1}$	a real spherical conic section $\nu = 0$	(130) (132) (131)
$\sqrt{k_1 - \dfrac{2}{\tan \bar{a}_0}}$ / $2 \tan \bar{a}_0 + k_1$	$\tanh \lvert a_1\rvert = \dfrac{\tan \alpha_1}{\tan \bar{a}_0}$	$\lvert\varkappa_0\rvert = \dfrac{90^\circ - \bar{a}_0}{\lvert a_1\rvert}$	an imaginary spherical hyperbola $\nu = 0$	(192) (194) (193)
$= 0$	$\sinh \lvert a_1\rvert = k_1$	$\lvert\varkappa_0\rvert = 0$	[7]) an imaginary spherical conic section having $e = 0$ $\lvert\nu\rvert = 0$ (Spherical helix)	(213) (214) —
$= 0$	$a_1 = 90^\circ$	$\varkappa_\nu = \dfrac{a_\nu}{90^\circ}$	a spherical circle	(135)

Relations between φ_L and φ_R, and the angle between the axes of rotation for different rolling cases

$-180° < C_{L-R} < 180°$, $0 \leq a_\nu \leq 90°$, $0 \leq r_\nu \leq 180°$

Both curves must have the same value of a_1 and α_1. C_{L-R} is assumed to be positive in the direction from the axis of rotation of the L-curve to the point of contact of the rolling curves if r_L is positive and vice versa.

Rolling curves to the real spherical conic section	Rolling curves to the imaginary spherical ellipse	Rolling curves to the imaginary spherical hyperbola	Rolling curves to the imaginary spherical conic section if $e \to 0$
$r - a_\nu = r - \varkappa_\nu$ a_1 lies in the first or fourth quadrant for the main curve ($\tan \Phi$ then has the same sign as $\cos \varphi$) and in the second or third quadrant for the antipode curve ($\tan \Phi$ then has the opposite sign to $\cos \varphi$).	$r - a_\nu = r - \lvert\varkappa_\nu\rvert \lvert a_1\rvert$ lies in the same quadrant as φ_ν for the main curve ($\tan \Phi$ is then positive) and in the same quadrant as $\varphi_\nu + 180°$ for the antipode curve ($\tan \Phi$ is then negative).	$r_\nu - \bar{a}_\nu = r_\nu - (90° - \lvert\varkappa_\nu\rvert \lvert a_1\rvert)$ lies in the first or fourth quadrant for the main curve ($\tan \Phi$ then has the same sign as $\cos \varphi$) and in the second or third quadrant for the antipode curve ($\tan \Phi$ then has the opposite sign to $\cos \varphi$).	$r - a_\nu = r - \lvert\varkappa_\nu\rvert \lvert a_1\rvert$ lies in the same quadrant as φ_ν for the main curve ($\tan \Phi$ is then positive) and in the same quadrant as $\varphi_\nu + 180°$ for the antipode curve ($\tan \Phi$ is then negative).
Rolling case 1: $\varphi_R = \varphi_L$ where both curves are main curves or both antipode curves			
$C_{L-R} = a_{\nu L} - a_{\nu R} = a_1 (\varkappa_L - \varkappa_R)$	$C_{L-R} = a_{\nu L} - a_{\nu R} = \lvert a_1\rvert (\lvert\varkappa_L\rvert - \lvert\varkappa_R\rvert)$	$C_{L-R} = \bar{a}_{\nu L} - \bar{a}_{\nu R} = \lvert a_1\rvert (\lvert\varkappa_R\rvert - \lvert\varkappa_L\rvert)$	$C_{L-R} = a_{\nu L} - a_{\nu R} = \lvert a_1\rvert (\lvert\varkappa_L\rvert - \lvert\varkappa_R\rvert)$
Rolling case 2: $\varphi_R = -\varphi_L$ where both curves are main curves or both antipode curves			
$C_{L-R} = a_{\nu L} + a_{\nu R} = a_1 (\varkappa_L + \varkappa_R)$	$C_{L-R} = a_{\nu L} + a_{\nu R} = \lvert a_1\rvert (\lvert\varkappa_L\rvert + \lvert\varkappa_R\rvert)$	$C_{L-R} = \bar{a}_{\nu L} + \bar{a}_{\nu R} = 180° - \lvert a_1\rvert (\lvert\varkappa_R\rvert + \lvert\varkappa_L\rvert)$	$C_{L-R} = a_{\nu L} + a_{\nu R} = \lvert a_1\rvert (\lvert\varkappa_L\rvert + \lvert\varkappa_R\rvert)$
Rolling case 3: $\varphi_R = 180° - \varphi_L$			
One of the curves is a main curve, the other an antipode curve	Both curves are main curves or both antipode curves	One of the curves is a main curve, the other an antipode curve	Both curves are main curves or both antipode curves
$C_{L-R} = a_{\nu L} - a_{\nu R} \mp 180° = a_1 (\varkappa_L - \varkappa_R) \mp 180°$	$C_{L-R} = a_{\nu L} + a_{\nu R} - 180° = \lvert a_1\rvert (\lvert\varkappa_L\rvert + \lvert\varkappa_R\rvert) - 180°$	$C_{L-R} = \bar{a}_{\nu L} - \bar{a}_{\nu R} \mp 180° = \lvert a_1\rvert (\lvert\varkappa_R\rvert - \lvert\varkappa_L\rvert) \mp 180°$	$C_{L-R} = a_{\nu L} + a_{\nu R} - 180° = \lvert a_1\rvert (\lvert\varkappa_L\rvert + \lvert\varkappa_R\rvert) - 180°$
Rolling case 4: $\varphi_R = 180° + \varphi_L$			
One of the curves is a main curve, the other an antipode curve	Both curves are main curves or both antipode curves	One of the curves is a main curve, the other an antipode curve	Both curves are main curves or both antipode curves
$C_{L-R} = a_{\nu L} + a_{\nu R} - 180° = a_1 (\varkappa_L + \varkappa_R) - 180°$	$C_{L-R} = a_{\nu L} - a_{\nu R} \mp 180° = \lvert a_1\rvert (\lvert\varkappa_L\rvert - \lvert\varkappa_R\rvert) \mp 180°$	$C_{L-R} = \bar{a}_{\nu R} + \bar{a}_{\nu L} - 180° = -\lvert a_1\rvert (\lvert\varkappa_R\rvert + \lvert\varkappa_L\rvert)$	$C_{L-R} = a_{\nu L} - a_{\nu R} \mp 180° = \lvert a_1\rvert (\lvert\varkappa_L\rvert - \lvert\varkappa_R\rvert) \mp 180°$

Derivation of the mating curve (*R*-curve) to a given spherical conic section rolling curve (*L*-curve) for a given angle between the axes of rotation

Find from App. 3 the constants $\varkappa_R$ and $a_{\nu R}$ or $|\varkappa_R|$ and $a_{\nu R}$ or $a_{\nu R}$ for the mating curve and also the relation between φ_L and φ_R and if the mating curve is the main or antipode curve. As $a_{\nu R} > 0$ and $< 90°$, one solution only is obtained. The sign of C_{L-R} is according to the explanations in App. 3. The same values of a_1 and α_1 apply to both curves.

For the *R*-curve the following is obtained if the curves concerned are rolling curves to

a real spherical conic section $\sin \alpha_1 = \tan e\, a_1 / \tan a_1$				an imaginary spherical ellipse $\sin \alpha_1 = \tanh e\, \lvert a_1 \rvert / \tanh \lvert a_1 \rvert$						an imaginary spherical hyperbola $\tan \alpha_1 = \tanh \lvert a_1 \rvert / \tan \lvert e \rvert \lvert a_1 \rvert$				an imaginary spherical conic section where $e \to 0$			
				$\alpha_1 \neq 90°$			$\alpha_1 = 90°$										
If for the *R*-curve	a_ν and ν or $\lvert\nu\rvert$	eq. in parametric form	eq. in non-parametric form	If for the *R*-curve	a_ν and $\lvert\nu\rvert$	eq. in parametric form	eq. in non-parametric form	a_ν and ν	eq. in parametric form	eq. in non-parametric form	If for the *R*-curve	a_ν and ν	eq. in parametric form	eq. in non-parametric form	If for the *R*-curve	a_ν and $\lvert\nu\rvert$	eq. in parametric form	eq. in non-parametric form
	from eq.				from eq.							from eq.				from eq.		
$90° > \varkappa_\nu > e$, i.e. $\infty > \tan a_\nu > \tan a_1 \sin \alpha_1$	(108)	(107) or (135) if $\alpha_1 = 0$	(109)								$\lvert\varkappa_\nu\rvert < \lvert e \rvert$ i.e. $\tan \bar{a}_\nu < \frac{\tanh \alpha_1}{\tan \lvert a_1 \rvert}$	(172)	(171)	(174)				
$\varkappa_\nu < e$ i.e. $\tan a_\nu < \tan a_1 \sin \alpha_1$	(116)	(115)	(117)	$0 < \lvert\varkappa_\nu\rvert < \frac{90°}{\lvert a_1 \rvert}$ i.e. $0 < \tan a_\nu < \infty$	(145)	(144)	(146)	(159) $\lvert\nu_R\rvert = \lvert\nu_L\rvert$	(158)	(160) or (162)	$\frac{90°}{\lvert a_1 \rvert} > \lvert\varkappa_\nu\rvert > \frac{\lvert e \rvert}{\lvert a_1 \rvert}$ i.e. $\tanh \alpha_1 / \tan \lvert a_1 \rvert > \tan \bar{a}_\nu > 0$	(178)	(177)	(179)	$\frac{90°}{\lvert a_1 \rvert} > \lvert\varkappa_\nu\rvert > 0$	(206)	(205)	(207)
$\varkappa_\nu = e$ i.e. $\nu = 0$ and $\tan a_\nu = \tan a_1 \sin \alpha_1$	(131)	(130)	(132)								$\lvert\varkappa_\nu\rvert = \frac{\lvert e \rvert}{\lvert a_1 \rvert}$ i.e. $\tan \bar{a}_\nu = \frac{\tanh \alpha_1}{\tan \lvert a_1 \rvert}$	(193)	(192)	(194)	$\lvert\varkappa_\nu\rvert = e = 0$		$a_\nu = 0,\ \lvert\nu\rvert = 0$	(213)	(214)
$\varkappa_\nu = 0$ i.e. $\tan a_\nu = 0$	(126)	(125)	(127)	$\lvert\varkappa_\nu\rvert = 0$ i.e. $\tan a_\nu = 0$	(154)	(153)	(155)				$\lvert\varkappa_\nu\rvert = \frac{90°}{\lvert a_1 \rvert}$ i.e. $\tan \bar{a}_\nu = 0$	(188)	(187)	(189)				
$\varkappa_\nu = \frac{90°}{a_1}$ i.e. $\tan a_\nu = \infty$	(121)	(120)	(122)	$\lvert\varkappa_\nu\rvert = \frac{90°}{\lvert a_1 \rvert}$ i.e. $\tan a_\nu = \infty$	(149)	(148)	(150)				$\lvert\varkappa_\nu\rvert = 0$ i.e. $\tan \bar{a}_\nu = \infty$	(183)	(182)	(184)	$\lvert\varkappa_\nu\rvert = \frac{90°}{\lvert a_1 \rvert}$	(210)	(209)	(211)

The ranges of φ for the different branches with imaginary ν

Larger branch [1]	Smaller branch [2]	tan-branch [1]	cot-branch [2]	Larger branch [1]	Smaller branch [2]	Branches [3] tan- A	tan- B	cot- A	cot- B
$-\alpha_\nu \leq \varphi_\nu \leq 180° + \alpha_\nu$	$180° + \alpha_\nu \leq \varphi_\nu \leq -\alpha_\nu$	$\alpha_\nu - 90° \leq \varphi_\nu \leq \alpha_\nu + 90°$	$\alpha_\nu + 90° \leq \varphi_\nu \leq \alpha_\nu - 90°$	$-\alpha_\nu \leq \varphi_\nu \leq 180° + \alpha_\nu$	$180° + \alpha_\nu \leq \varphi_\nu \leq -\alpha_\nu$	$-\alpha_\nu \leq \varphi_\nu \leq 180°$	$0 \leq \varphi_\nu \leq -\alpha_\nu$	$180° - \alpha_\nu \leq \varphi_\nu \leq 180°$	$-\alpha_\nu \leq \varphi_\nu \leq 180°$

[1]) Upper expressions and upper signs in the equations

[2]) Lower » » lower » » »

[3]) tan-branches A: Upper sign for e, upper sign for $|\nu|$ and ϑ_0

» » B: Lower » » », lower » » »

cot- » A: » » » », upper » » »

» » B: Upper » » », lower » » » and ϑ_0

If ν is imaginary the relation between φ_R and φ_L and the boundary values given above define the kind of branch which meshes for the value of φ_L in question.

For external gears the result thus obtained should be compared with the tables on pp. 64, 76, 90 and 99.

Evaluating the angle Φ from the radius vector to the normal

a) *Rolling curves to the real spherical conic section*

$$\tan \Phi = \pm \frac{\tan \alpha_1 \cos \varphi}{\cos a_1 \sqrt{1 + \tan^2 a_1 \sin^2 \alpha_1 \sin^2 \varphi}} = \frac{\tan \alpha_1 \cos (r_\nu - a_\nu) \cos \varphi}{\cos a_1}$$

If $a_1 = 90^\circ$ and $\alpha_1 = 0^\circ$, then $\tan a_1 \sin \alpha_1 = \dfrac{\tan \alpha_1}{\cos a_1} = \tan e\, a_1$.
The upper sign is valid for the main curve.
» lower » » » » » antipode curve.

b) *Rolling curves to the imaginary spherical ellipse*

$$\tan \Phi = \pm \frac{\tan \alpha_1}{\cosh |a_1| \sqrt{1 - (1 - \tanh^2 |a_1|) \sin^2 \alpha_1) \sin^2 \varphi}} = \frac{\tan \alpha_1 \cos (r_\nu - a_\nu)}{\cosh |a_1| \cos \varphi}$$

If $|a_1| = \infty$ and $\alpha_1 = 90^\circ$, then $\tan \Phi = \pm |\nu|$.
The upper signs are valid for the main curve.
» lower » » » » » antipode curve.

c) *Rolling curves to the imaginary spherical hyperbola*

$$\tan \Phi = \pm \frac{\sin \alpha_1 \cos \varphi}{\sinh |a_1| \sqrt{1 + \dfrac{\tan^2 \alpha_1}{\tanh^2 |a_1|} \sin^2 \varphi}} = \frac{\sin \alpha_1 \cos (r_\nu - \bar{a}_\nu) \cos \varphi}{\sinh |a_1|}$$

The upper sign is valid for the main curve.
» lower » » » » » antipode curve.

d) *Rolling curves to the imaginary spherical conic sections when* $e \to 0$

$$\tan \Phi = \pm \frac{1}{\sqrt{\dfrac{\cosh^2 |a_1|}{\sin^2 \varphi} - 1}} = \pm \frac{|\sin (r_\nu - a_\nu)|}{\sinh |a_1|}$$

The upper signs are valid for the main curve.
» lower » » » » » antipode curve.

The relation between the angles of displacement for the curves in a

The relation between the polar angles ϑ_R and ϑ_L to the point of contact (the pitch point) for the two wheels in a spherical conic section rolling curve pair is obtained from the equations below.

a) *Rolling curves to the real spherical conic section and the imaginary spherical hyperbola* Line

$$\left.\begin{matrix}\tan \\ \cot\end{matrix}\right\} \nu_R \frac{\vartheta_R}{2}$$
$$\left.\begin{matrix}\tanh \\ \coth\end{matrix}\right\} |\nu_R| \frac{\vartheta_R}{2}$$
$$\left.\begin{matrix}\coth \\ \tanh\end{matrix}\right\} |\nu_R| \frac{\vartheta_R}{2}$$
$$\left.\begin{matrix}\vartheta_R c_R \\ \frac{1}{\vartheta_R c_R}\end{matrix}\right\}$$
$$\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \tan \frac{90^\circ \pm \alpha_{\nu R}}{2} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} = \left\{\begin{matrix} \pm \tan \frac{90^\circ + \alpha_{\nu L}}{2} \left\{\begin{matrix}\tan \nu_L \frac{\vartheta_L}{2} & (1) \\ \tanh |\nu_L| \frac{\vartheta_L}{2} & (2) \\ \coth |\nu_L| \frac{\vartheta_L}{2} & (3)\end{matrix}\right. \\ \pm c_L \vartheta_L \quad (4)\end{matrix}\right.$$

where

for rolling curves to the real spherical conic section

$$c_R = \frac{\sin a_{\vartheta R}}{\sin a_1 \cos \alpha_1}, \; c_L = \frac{\sin a_{\vartheta L}}{\sin a_1 \cos \alpha_1}$$

for rolling curves to the imaginary spherical hyperbola

$$c_R = \frac{\sin \bar{a}_{\vartheta R} \cos \alpha_1}{\cosh |a_1|}, \; c_L = \frac{\sin \bar{a}_{\vartheta L} \cos \alpha_1}{\cosh |a_1|}$$

For rolling curves of the above type the following is valid:

Line 1 for rolling curves with real ν-values
» 2 for the larger branch of curves with imaginary ν-values
» 3 for the smaller branch of curves with imaginary ν-values
» 4 for curves with $\nu = 0$

Any line on the left-hand side of the equation may be combined with any line on the right-hand side. For the different cases of rolling the following is valid

Case of rolling No.	$\varphi_R =$	In the left-hand side of the equation	In the right-hand side of the equation
1	φ_L	Upper sign and upper expression	Upper sign
2	$-\varphi_L$	Lower sign and lower expression	Upper sign
3	$180^\circ - \varphi_L$	Upper sign and upper expression	Lower sign
4	$180^\circ + \varphi_L$	Lower sign and lower expression	Lower sign

b) *Rolling curves to the imaginary spherical ellipse*

$$\pm [\alpha_{\nu R} \mp (\pm 90^\circ + \operatorname{amh} |\nu_R| \vartheta_R)] = \alpha_{\nu L} \mp (90^\circ + \operatorname{amh} |\nu_L| \vartheta_L)$$

$$(\operatorname{amh} |\nu| \vartheta = 2 \arctan e^{|\nu| \vartheta} - 90^\circ)$$

rolling curve pair consisting of spherical conic section rolling curves

The signs are chosen according to the rules in the tables below.

Branch	Before $(\pm 90° + \operatorname{amh} \lvert \nu_R \rvert \vartheta_R)$ and $(90° + \operatorname{amh} \lvert \nu_R \rvert \vartheta_R)$ use
tan-branch	Upper sign
cot-branch	Lower sign

Case of rolling No.	$\varphi_R =$	For the left-hand side of the equation use	For $\pm 90°$
1	φ_L	Upper sign	Upper sign
2	$-\varphi_L$	Lower sign	Lower sign
3	$180° + \varphi_L$	Lower sign	Upper sign
4	$180° - \varphi_L$	Upper sign	Lower sign

c) *Rolling curves to the imaginary conic sections when* $e \to 0$

Line

$$\left.\begin{aligned} &\pm\left[\frac{1}{\tan\,\alpha_{\nu R}}\left(\frac{\pm\, e^{\pm \lvert \nu_R \rvert \vartheta_R}}{\cos^2 a_{\nu R}} - 1\right)\right] \\ &\pm\left(\pm\, e^{\pm \lvert \nu_R \rvert \vartheta_R} \lvert \tan R_{zR} \rvert \tanh \lvert a_1 \rvert\right) \\ &\pm \frac{\pm\, \vartheta_R}{\cosh \lvert a_1 \rvert} \end{aligned}\right\} = \left\{\begin{aligned} &\frac{1}{\tan\,\alpha_{\nu L}}\left(\frac{\pm\, e^{\pm \lvert \nu \rvert \vartheta_L}}{\cos^2 a_{\nu L}} - 1\right) && (1) \\ &\pm\, e^{\pm \lvert \nu_L \rvert \vartheta_L} \lvert \tan R_{zL} \rvert \tanh \lvert a_1 \rvert && (2) \\ &\frac{\pm\, \vartheta_L}{\cosh \lvert a_1 \rvert} && (3) \end{aligned}\right.$$

For rolling curves of this type the following is valid:

Line 1 for normal rolling curves
» 2 for crown-wheel curves
» 3 for curves with $\lvert \nu \rvert = 0$

Any line on the left-hand side of the equation may be combined with any line on the right-hand side. The signs are chosen according to the rules in the tables below.

Branch	Sign for e	Sign for $\lvert \nu \rvert \vartheta$ or ϑ
tan-A	Upper	Upper
tan-B	Lower	Lower
cot-A	Lower	Upper
cot-B	Upper	Lower

Case of rolling No.	$\varphi_R =$	For the left-hand side of the equation use
1	φ_L	Upper sign
2	$-\varphi_L$	Lower sign
3	$180° - \varphi_L$	Lower sign
4	$180° + \varphi_L$	Upper sign

$$\cos^2 a_\nu = \cos^2 \lvert \varkappa_\nu \rvert \lvert a_1 \rvert = (\lvert \nu \rvert^2 + 1) \cos^2 \alpha_\nu$$

To obtain the relation between the angles of displacement from a given starting position, the initial angle for one wheel, e.g. $\vartheta_L = (\vartheta_L)_s$, is first substituted in the equation. Then the correlated angle $\vartheta_R = (\vartheta_R)_s$ for the other wheel can be evaluated. Next, $\vartheta_R = \gamma_R + (\vartheta_R)_s$ and $\vartheta_L = \gamma_L + (\vartheta_L)_s$ are substituted in the equation, which gives the relation required. It is convenient to choose the starting position at the point of contact (the pitch point) corresponding to $\vartheta_L = 0$ or $\vartheta_R = 0$. This is, however, only possible if the pitch curves mesh at this point. In the opposite case, the starting point for γ may be chosen at one end of the arc used.

For external gears γ_R and γ_L obtain opposite signs and for internal gears they obtain the same signs.

Evaluating the arc length of

The formulae give the arc length S in radians.

a) *Rolling curves to the real spherical conic section*

$S = S_e$ is measured from the point where $\varphi = 0$

$$dS = \tan a_1 \sqrt{1 - \sin^2 a_1 \cos^2 \alpha_1} \cdot \frac{\sqrt{1 - \sin^2 \alpha_1 \sin^2 \varphi}}{1 + \tan^2 a_1 \sin^2 \alpha_1 \sin^2 \varphi}\, d\varphi$$

$$S = \sqrt{\frac{(1-p)(p - \sin^2 \alpha_1)}{p}} \cdot \Pi(p, \alpha_1, \varphi) + \arctan\left(\frac{\tan a_1 \sin^2 \alpha_1}{\sqrt{1 - \sin^2 a_1 \cos^2 \alpha_1}} \cdot \frac{\sin \varphi \cos \varphi}{\sqrt{1 - \sin^2 \alpha_1 \sin^2 \varphi}}\right)$$

where $p = \dfrac{\sin^2 \alpha_1}{1 - \sin^2 a_1 \cos^2 \alpha_1}$

or if $a_1 = 90°$, $\alpha_1 = 0$

$$S = \arctan \frac{\tan \varphi}{\cos e\, a_1}$$

The arc length of a quadrant (e.g. from $\varphi = 0$ to $\varphi = 90°$) is

$$S_{1/4} = F(\alpha_1) \cdot E[(90° - \alpha_1), a_1] - [F(\alpha_1) - E(\alpha_1)] \cdot F[(90° - \alpha_1), a_1] = \frac{\pi}{2} \Lambda_0(\alpha_1, a_1)$$

By increasing ϑ { For curves with real ν-values and the larger branch of curves with imaginary ν-values: S and φ decrease
For the smaller branch of curves with imaginary ν-values: S and φ increase

b) *Rolling curves to the imaginary spherical ellipse*

$S = S_{ei}$ is measured from the point where $\varphi = 0$

$$dS = \sinh |a_1| \sqrt{1 - \tanh^2 |a_1| \sin^2 \alpha_1} \cdot \frac{\sqrt{1 - \cos^2 \alpha_1 \sin^2 \varphi}}{1 - (1 - \tanh^2 |a_1| \sin^2 \alpha_1) \sin^2 \varphi}\, d\varphi$$

$$S = \sqrt{\frac{(1-p)(p - \cos^2 \alpha_1)}{p}} \cdot \Pi[p, (90° - \alpha_1), \varphi] + \sqrt{\frac{1-p}{p(p - \cos^2 \alpha_1)}} \cos^2 \alpha_1 \cdot F[(90° - \alpha_1), \varphi]$$

where $p = 1 - \tanh^2 |a_1| \sin^2 \alpha_1$

or if $\alpha_1 = 90°$, $|a_1| = \infty$

$$S = \sqrt{1 + \frac{1}{|\nu|^2}} \cdot \varphi$$

The arc length of a quadrant (e.g. from $\varphi = 0$ to $\varphi = 90°$) is

$$S_{1/4} = F(90° - \alpha_1) \cdot E(\alpha_1, \beta) - [F(90° - \alpha_1) - E(90° - \alpha_1)] \cdot F(\alpha_1, \beta) +$$

$$+ \frac{\sinh |a_1| \cos^2 \alpha_1}{\sqrt{1 - \tanh^2 |a_1| \sin^2 \alpha_1}} \cdot F(90° - \alpha_1) = \frac{\pi}{2} \Lambda_0(90° - \alpha_1, \beta) + \frac{\sinh |a_1| \cos^2 \alpha_1}{\sqrt{1 - \tanh^2 |a_1| \sin^2 \alpha_1}} F(90° - \alpha_1)$$

where $\tan \beta = \dfrac{1}{\cos \alpha_1 \sinh |a_1|}$

By increasing ϑ { For the tan-branches: φ and S decrease
For the cot-branches: φ and S increase

spherical conic section rolling curves

c) *Rolling curves to the imaginary spherical hyperbola*

$S = S_{hi}$ is measured from the point where $\varphi = 90°$

$$dS = -\frac{\sinh|a_1|}{\sin\alpha_1\sqrt{1+\frac{\tanh^2|a_1|}{\tan^2\alpha_1}}}\cdot\frac{\sqrt{1-\sin^2\alpha_1\cos^2\varphi}}{1-\frac{1}{1+\frac{\tanh^2|a_1|}{\tan^2\alpha_1}}\cos^2\varphi}\,d(90°-\varphi)$$

$$S=\sqrt{\frac{(1-p)(p-\sin^2\alpha_1)}{p}}\cdot\Pi[p,\alpha_1,(90°-\varphi)]+\sqrt{\frac{1-p}{p(p-\sin^2\alpha_1)}}\sin^2\alpha_1\cdot F[\alpha_1,(90°-\varphi)]$$

where $p=\dfrac{1}{1+\dfrac{\tanh^2|a_1|}{\tan^2\alpha_1}}$

The arc length of a quadrant (e.g. from $\varphi = 90°$ to $\varphi = 0$) is

$$S_{1/4}=F(\alpha_1)\cdot E[(90°-\alpha_1),\beta]-[F(\alpha_1)-E(\alpha_1)]\cdot F[(90°-\alpha_1),\beta]+$$

$$+\tanh|a_1|\sqrt{\sin^2\alpha_1+\sinh^2|a_1|}\cdot F(\alpha_1)=\frac{\pi}{2}\Lambda_0(\alpha_1,\beta)+\tanh|a_1|\sqrt{\sin^2\alpha_1+\sinh^2|a_1|}\cdot F(\alpha_1)$$

where $\tan\beta=\dfrac{1}{\sinh|a_1|}$ or $\beta = 90° - \operatorname{amh}|a_1|$

If in the expressions for the arc length of the imaginary spherical ellipse

α_1 is replaced by $90° - \alpha_1$
φ » » » $90° - \varphi$
$\sinh|a_1|$ » » » $\dfrac{\sinh|a_1|}{\sin\alpha_1}$

the above expressions for the arc length of the imaginary spherical hyperbola are obtained.

By increasing ϑ { For curves with real v-values and the larger branch of curves with imaginary v-values: φ decreases and S increases
For the smaller branch of curves with imaginary v-values: φ increases and S decreases

d) *Rolling curves to the imaginary spherical conic sections when* $e \to 0$

$S = S_{exp}$ is measured from the point where $\varphi = 90°$ or $\varphi = 270°$

$$S=\pm\left[\sinh|a_1|\cdot\ln\left|\tan\frac{\varphi}{2}\right|-\operatorname{arc}\tan\frac{\cos\varphi}{\sinh|a_1|}\right]$$

The upper sign is valid for the A-branches
» lower » » » » » B- »

By increasing ϑ { For the tan-branches: φ and S decrease
For the cot-branches: φ and S increase

Appendix 8

Determining the out-of-centre setting of the cutter when cutting bevel gear teeth with milling cutters

b_{so} = out-of-centre setting of the cutter (to the left when cutting the right-hand face of the tooth space and vice versa)

c_b = bottom clearance

d = whole depth of tooth space, stamped on the milling cutter = $2\,m + c_b$

g_p = chord thickness of cutter used, measured at pitch line

h_{pc} = calliper setting of height when measuring chord thickness of cutter at pitch line

m = module

Z_c = number of teeth for which the cutter is designed when milling cylindrical gears

ψ = pressure angle

g_p may be measured with a gear-tooth vernier using a calliper setting of height $h_{pc} = d - m\,c_{pc}$ where $c_{pc} = 1 + \frac{Z_c}{2}\left(1 - \cos\frac{90^\circ}{Z_c}\right)$ is taken from the table below

Z_c	c_{pc}	Z_c	c_{pc}	Z_c	c_{pc}
12	1.0513	21	1.0294	42	1.0147
13	1.0474	23	1.0268	55	1.0112
14	1.0440	26	1.0237	80	1.0077
15	1.0411	30	1.0206	135	1.0046
17	1.0362	35	1.0176	∞	1.0000
19	1.0324				

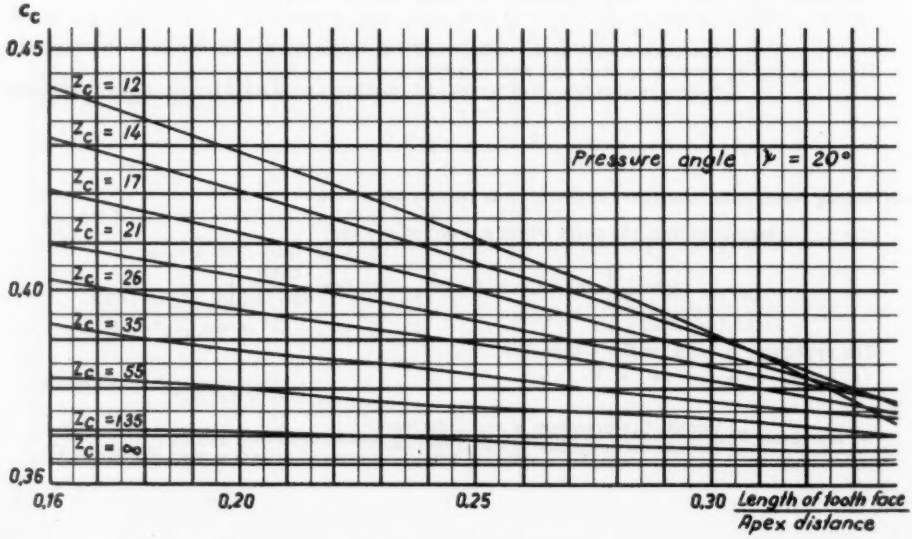
Values of c_c for $\psi = 20^\circ$

Z_c	Length of tooth face/Apex distance					
	1/3	1/3.5	1/4	1/4.5	1/5	1/6
12	0.376	0.397	0.411	0.421	0.429	0.440
14	0.379	0.395	0.406	0.414	0.421	0.430
17	0.379	0.391	0.400	0.407	0.412	0.419
21	0.378	0.387	0.394	0.399	0.403	0.409
26	0.376	0.384	0.389	0.393	0.396	0.401
35	0.374	0.379	0.383	0.386	0.388	0.392
55	0.371	0.374	0.376	0.378	0.380	0.382
135	0.367	0.368	0.369	0.370	0.371	0.371
∞	0.364	0.364	0.364	0.364	0.364	0.364

Finally we obtain

$$b_{so} = \frac{g_p}{2} - m\,c_c$$

If $\psi = 20^\circ$, c_c is taken from the curves below or the table above



Cutting circular bevel gears with disc milling cutters

b_{so} = out-of-centre setting of the cutter when cutting the side cuts

c_b = bottom clearance

c_c = constant to be taken from Appendix 8

D_y = outer diameter of the blank

d = $2\,m + c_b$ = whole depth of tooth space when cutting spur gears, usually stamped on the cutter

F = length of tooth face

h_{cC} = cutting depth at the large end of the tooth

m = module

R_{sph} = length of the generatrix of the pitch cone

V = cutting angle (tilting of the arbor of the dividing head from the horizontal position)

Z = number of teeth

Z_c = number of teeth for which the cutter is designed

α_p = half the apex angle of the pitch cone

Δv_{dC} = additional rotation of the arbor of the dividing head when cutting the side cuts. The rotation is so chosen that the tooth space is increased.

$$D_y = 2\,[R_{sph} \sin \alpha_p + m \cos \alpha_p]$$

$Z_c \leq \dfrac{Z}{\cos \alpha_p}$ The nearest lower standard value of Z_c is chosen

$$\sin \alpha_p = \frac{m\,Z}{2\,R_{sph}}$$

$$V = \alpha_p - B_{th} + \Delta V$$

where

$$\sin B_{th} = \frac{m}{R_{sph}} = \frac{2 \sin \alpha_p}{Z}$$

$$\Delta V = \frac{m}{R_{sph}} \left[\frac{c_c\,\pi}{2\,Z_c} - \frac{c_c^2}{Z} \cos \alpha_p \right] \frac{180^\circ}{\pi}$$

$$h_{cC} = d - m \left[\frac{\pi^2}{16\,Z_c} - \frac{c_c^2 \cos V}{Z} + \frac{2 \sin^2 \alpha_p}{Z^2} \right]$$

$$\Delta v_{dC} = \pm \frac{90^\circ}{Z} \left(1 - \frac{4\,c_c}{\pi} \right)$$

b_{so} is obtained from Appendix 8.

Formulae for determining the settings for dividing head, milling cutters when cutting

Erect the rotary milling table, the dividing head and the wheel blank on the milling-machine table as shown in Fig. 13. The index plates should be graduated in degrees.

A = addendum in angular measure, taken from eqs. (261)

$B_{th} = \text{arc sin} \dfrac{m}{R_{sph}}$ = dedendum minus bottom clearance in angular measure

c as a suffix indicates that the value refers to the tooth space or the bottom curve

D = diameter of setting disc

f = distance from the centre of the dividing head to the apex

h = height setting of milling-machine table

l = cross-setting of milling-machine table

m = module

R_{sph} = length of the generatrix of the pitch cone

r = spherical radius vector

t as a suffix indicates that the value refers to the tooth tip and top curve

v_d = the indexing of the dividing head (angle of rotation of the arbor of the dividing head)

v_h = setting of the rotary milling table

v_v = tilting from the horizontal position of the arbor of the dividing head

Θ = polar angle

Φ = the angle from the spherical radius vector to the normal

Values without suffix refer to the pitch curve.

Formulae for the settings when cutting the central cuts of the tooth spaces

$$\cos r_c = \cos B_{th} \cos r + \sin B_{th} \sin r \cos \Phi$$

$$\sin \Phi_c = \frac{\sin r}{\sin r_c} \sin \Phi$$

$$\sin v_{vc} = \sin r_c \cos \Phi_c$$

$$\tan v_{hc} = \tan r_c \sin \Phi_c$$

$$\sin (\Theta - \Theta_c) = \frac{\sin B_{th}}{\sin r} \sin \Phi_c$$

$$\tan (v_{dc} - \Theta_c) = \frac{\tan \Phi_c}{\cos r_c}$$

$$v_{dc} = (v_{dc} - \Theta_c) - (\Theta - \Theta_c) + \Theta$$

$$l_c = f \cos v_{vc} \sin v_{hc}$$

$$h_c = \frac{D_c}{2} - 2m - f \sin v_{vc}$$

D_c may be chosen arbitrarily, but is conveniently chosen as

$$D_c = 2 [2m + f \sin (v_{vc})_{\max}]$$

Formulae for the settings when machining the contour of the blank

$$\cos r_t = \cos A \cos r - \sin A \sin r \cos \Phi$$

$$\sin \Phi_t = \frac{\sin r}{\sin r_t} \sin \Phi$$

$$\sin v_{vt} = \sin r_t \cos \Phi_t$$

$$\tan v_{ht} = \tan r_t \sin \Phi_t$$

$$\sin (\Theta_t - \Theta) = \frac{\sin A}{\sin r} \sin \Phi_t$$

$$\tan (v_{dt} - \Theta_t) = \frac{\tan \Phi_t}{\cos r_t}$$

$$v_{dt} = (v_{dt} - \Theta_t) + (\Theta_t - \Theta) + \Theta$$

$$l_t = f \cos v_{vt} \sin v_{ht}$$

$$h_t = \frac{D_t}{2} - f \sin v_{vt}$$

D_t may be chosen arbitrarily, but is conveniently chosen as

$$D_t = 2 f \sin (v_{vt})_{\max}$$

The calculations are carried out for one period of the wheel or for half a period if the wheel is symmetrical.

rotary milling table, milling-machine table and the selection of non-circular bevel gears

When milling the side cuts, the settings v_{hc} and l_c are changed to:

For the left-hand cuts

$$v_{hcL} = v_{hc} - \Delta v_{hc}$$

$$l_{cL} = f \sin (v_{hc} - \Delta v_{hc}) \cos v_{vc} + b_{s0}$$

For the right-hand cuts

$$v_{hcR} = v_{hc} + \Delta v_{hc}$$

$$l_{cR} = f \sin (v_{hc} + \Delta v_{hc}) \cos v_{vc} - b_{s0}$$

where

$$\Delta v_{hc} = \frac{m}{R_{sph} \cos B_{lh}} \left(45^\circ - c_c \frac{180^\circ}{\pi} - \frac{\pi^2 \, 15^\circ}{8 \, Z_c^2} \right)$$

and c_c and b_{s0} are taken from Appendix 8.

If a greater accuracy is desired, the value B_{lh} in the preceding equations is increased by an amount ΔB_{lh} and the value h_c is reduced by an amount Δh_c, where ΔB_{lh} and Δh_c are obtained from eqs. (273), (274) and (275). Generally, however, these corrections will be insignificant.

The above rules are valid provided that increasing the settings gives a clockwise rotation of the arbor of the dividing head as viewed from the dividing head, and of the rotary milling table as viewed from above, and further that the milling table is elevated or moved to the left as viewed from the dividing head.

To express the angles v_d and v_h in revolutions + degrees on the index plates, divide v, expressed in degrees, by $\frac{360}{\text{worm-gear ratio}}$. The integral quotient gives the number of revolutions. Multiplying the remainder by the worm-gear ratio gives the degrees.

Selection of milling cutters

Choose a standard cutter of the nearest lower value of $(Z_c)_n$ obtained from

$$(Z_c)_n \leq \frac{360^\circ}{\Delta\Phi + \Delta\vartheta \cos (\gamma)_n}$$

where

Z_c = the number of teeth for which the cutter is designed when milling spur wheels

$\Delta\Phi = (\Phi)_{n+1/2} - (\Phi)_{n-1/2}$

$\Delta\vartheta = (\vartheta)_{n+1/2} - (\vartheta)_{n-1/2}$

ϑ = polar angle

n = suffix indicating that the value refers to tooth-space number n. Suffixes $n + \frac{1}{2}$ and $n - \frac{1}{2}$ refer to the adjacent teeth.

Zero positions for the settings

$v_v = 0$ corresponds to the horizontal position of the arbor of the dividing head

$v_d = 0$ when the radius for $\Theta = 0$ points upwards in the vertical plane

$v_h = 0$ when the arbor of the dividing head is parallel to the longitudinal direction of the milling table

l_c and $h_c = 0$, when $v_v = 0$ and $v_h = 0$ and the cutter is first set directly above the centre of the arbor, touching the setting disc and then the milling-machine table is elevated the distance $d = 2\,m$ + bottom clearance, usually marked on the cutter.

$l_t = 0$ and $h_t = 0$, when v_v and $v_h = 0$ and the cutter is set directly above the centre of the arbor, touching the setting disc.

In the above, it has been assumed that all the cutters are symmetrical and have the same diameter, and the same length of the boss.

Table showing the arc length S_φ as a function of φ for a real spherical conic section with $a = 29.670°$ and $\alpha = 30°$

φ	S_φ	ΔS_φ 10^5	φ	S_φ	ΔS_φ 10^5	φ	S_φ	ΔS_φ 10^5
0	0.00000	898	30	0.26472	851	60	0.50701	762
1	0.00898	898	31	0.27323	848	61	0.51463	760
2	0.01796	898	32	0.28171	846	62	0.52223	756
3	0.02694	898	33	0.29017	842	63	0.52979	755
4	0.03592	897	34	0.29859	840	64	0.53734	752
5	0.04489	897	35	0.30699	837	65	0.54486	749
6	0.05386	896	36	0.31536	833	66	0.55235	748
7	0.06282	895	37	0.32369	831	67	0.55983	745
8	0.07177	894	38	0.33200	828	68	0.56728	743
9	0.08071	893	39	0.34028	824	69	0.57471	741
10	0.08964	892	40	0.34852	822	70	0.58212	739
11	0.09856	891	41	0.35674	818	71	0.58951	737
12	0.10747	890	42	0.36492	815	72	0.59688	735
13	0.11637	888	43	0.37307	813	73	0.60423	734
14	0.12525	887	44	0.38120	809	74	0.61157	732
15	0.13412	885	45	0.38929	806	75	0.61889	730
16	0.14297	883	46	0.39735	803	76	0.62619	729
17	0.15180	882	47	0.40538	799	77	0.63348	728
18	0.16062	880	48	0.41337	797	78	0.64076	727
19	0.16942	878	49	0.42134	794	79	0.64803	725
20	0.17820	875	50	0.42928	790	80	0.65528	724
21	0.18695	874	51	0.43718	788	81	0.66252	724
22	0.19569	871	52	0.44506	784	82	0.66976	722
23	0.20440	869	53	0.45290	782	83	0.67698	722
24	0.21309	867	54	0.46072	779	84	0.68420	721
25	0.22176	864	55	0.46851	775	85	0.69141	721
26	0.23040	862	56	0.47626	773	86	0.69862	720
27	0.23902	859	57	0.48399	770	87	0.70582	720
28	0.24761	857	58	0.49169	768	88	0.71302	720
29	0.25618	854	59	0.49937	764	89	0.72022	719
30	0.26472		60	0.50701		90	0.72741	

Auxiliary table for the calculation of a non-circular bevel gear

Tooth space No.	Tooth No.	$\sin r_1$	$\cos r_1$	$\sin r_2$	$\cos r_2$	$\sin \Phi$	$\cos \Phi$	$\sin r_1 \cdot \cos \Phi$	$\sin r_2 \cdot \cos \Phi$
1		0.71410	0.70005	0.97126	0.23804	0.00000	1.00000	0.71410	0.97126
2		0.70384	0.71036	0.96769	0.25214	0.20535	0.97869	0.68884	0.94707
3		0.67472	0.73808	0.95677	0.29084	0.36577	0.93070	0.62796	0.89047
4		0.63037	0.77630	0.93812	0.34631	0.47051	0.88240	0.55623	0.82780
5		0.57476	0.81832	0.91171	0.41083	0.52930	0.84844	0.48765	0.77353
6		0.51156	0.85925	0.87821	0.47826	0.55246	0.83354	0.42641	0.73202
7		0.44450	0.89578	0.83916	0.54389	0.54489	0.83851	0.37272	0.70364
8		0.37786	0.92586	0.79717	0.60375	0.50489	0.86318	0.32616	0.68810
9		0.31699	0.94842	0.75632	0.65420	0.42467	0.90535	0.28699	0.68473
10		0.26887	0.96317	0.72244	0.69143	0.29266	0.95621	0.25710	0.69080
11		0.24161	0.97037	0.70267	0.71152	0.10614	0.99435	0.24024	0.69870
	1½	0.71151	0.70267	0.97037	0.24161	0.10614	0.99435	0.70749	0.96489
	2½	0.69143	0.72244	0.96318	0.26887	0.29266	0.95621	0.66115	0.92100
	3½	0.65420	0.75632	0.94843	0.31699	0.42467	0.90535	0.59228	0.85866
	4½	0.60375	0.79717	0.92586	0.37786	0.50489	0.86318	0.52114	0.79918
	5½	0.54390	0.83916	0.89578	0.44450	0.54489	0.83851	0.45607	0.75112
	6½	0.47826	0.87821	0.85925	0.51156	0.55246	0.83354	0.39865	0.71622
	7½	0.41083	0.91171	0.81832	0.57476	0.52930	0.84844	0.34856	0.69430
	8½	0.34631	0.93812	0.77629	0.63037	0.47051	0.88240	0.30558	0.68500
	9½	0.29084	0.95677	0.73808	0.67472	0.36577	0.93070	0.27068	0.68693
	10½	0.25214	0.96770	0.71036	0.70384	0.20535	0.97869	0.24677	0.69522
	11½	0.23804	0.97126	0.70005	0.71410	0.00000	1.00000	0.23804	0.70005

Schedule for the calculation of φ, r, Φ

$S_{1/4} = 0.727414$ radians
$a_1 = 29.670°$
$\alpha_1 = 30°$
$\varkappa_2 a_1 = 60.330°$
$\alpha_2 = 9.339°$
$\frac{S_{1/4}}{21/4} = 0.138555$

$\tan a_1 = 0.56970$
$\sin \alpha_1 = 0.50000$
$\tan a_1 \sin \alpha_1 = 0.28485$

$\cos a_1 = 0.86889$, $\frac{\tan \alpha_1}{\cos a_1} = 0.66447$
$\tan \alpha_1 = 0.57735$
$\tan \frac{90° - \alpha_1}{2} = 0.57735$
$\tan \frac{90° - \alpha_2}{2} = 0.84898$

[1] Tooth space No. n	[1] Tooth No. n	$S_n = \pm 0.727414 - (n-1)\frac{S_{1/4}}{21/4}$ radians	[2] φ degrees	[2] $\sin \varphi$	$\cos \varphi$	[2] $\tan (r_1 - a_1) = \tan (r_2 - \varkappa_2 a_1) = \tan a_1 \sin \alpha_1 \sin \varphi$	[2] $r_1 - a_1 = r_2 - \varkappa_2 a_1$ degrees	r_1 degrees	r_2 degrees	$\cos (r_1 - a_1) = \cos (r_2 - \varkappa_2 a_1)$
1	11½	± 0.72741	± 90.000	± 1.00000	0.00000	± 0.28485	± 15.899	45.569 13.771	76.229 44.431	0.96174
2	10½	± 0.58886	± 70.912	± 0.94502	0.32702	± 0.26919	± 15.066	44.736 14.604	75.396 45.264	0.96563
3	9½	± 0.45030	± 52.668	± 0.79514	0.60643	± 0.22650	± 12.762	42.432 16.908	73.092 47.568	0.97529
4	8½	± 0.31175	± 35.569	± 0.58169	0.81342	± 0.16569	± 9.408	39.078 20.262	69.738 50.922	0.98655
5	7½	± 0.17319	± 19.429	± 0.33263	0.94306	± 0.09475	± 5.413	35.083 24.257	65.743 54.917	0.99554
6	6½	± 0.03464	± 3.857	± 0.06727	0.99763	± 0.01916	± 1.098	30.768 28.572	61.428 59.232	0.99982
7	5½	∓ 0.10392	∓ 11.602	∓ 0.20111	0.97957	∓ 0.05729	∓ 3.279	26.391 32.949	57.051 63.609	0.99836
8	4½	∓ 0.24247	∓ 27.402	∓ 0.46023	0.88780	∓ 0.13110	∓ 7.469	22.201 37.139	52.861 67.799	0.99151
9	3½	∓ 0.38103	∓ 43.979	∓ 0.69440	0.71959	∓ 0.19780	∓ 11.189	18.481 40.859	49.141 71.519	0.98099
10	2½	∓ 0.51958	∓ 61.651	∓ 0.88007	0.47484	∓ 0.25069	∓ 14.073	15.597 43.743	46.257 74.403	0.96999
11	1½	∓ 0.65814	∓ 80.395	∓ 0.98599	0.16685	∓ 0.28086	∓ 15.688	13.982 45.358	44.642 76.018	0.96275

[1] Counted from the max. radius both for the single-period and the two-period wheel

[2] Upper signs refer to tooth spaces in both wheels
Lower » » » teeth in both wheels

and ϑ for a non-circular bevel gear

$\tan \Phi = \frac{\tan \alpha_1 \cos (\gamma_1 - a_1) \cos \varphi}{\cos a_1}$	Φ degrees	$\frac{90^\circ - \varphi}{2}$ degrees	$\tan \frac{90^\circ - \varphi}{2}$	$\tan \frac{\vartheta_1}{2} = \tan \frac{90^\circ - \alpha_1}{2} \cdot \tan \frac{90^\circ - \varphi}{2}$	$\frac{\vartheta_1}{2}$ degrees	ϑ_1 degrees	$\tan \vartheta_2 = \tan \frac{90^\circ - \alpha_2}{2} \cdot \tan \frac{90^\circ - \varphi}{2}$	ϑ_2 degrees
0.00000	0.000	0.000	0.00000	0.00000	0.000	0.000	0.00000	0.000
		90.000	∞	∞	90.000	180.000	∞	90.000
0.20982	11.850	9.544	0.16813	0.09707	5.544	11.088	0.14274	8.123
		80.456	5.94770	3.43390	73.764	147.528	5.04948	78.798
0.39299	21.455	18.666	0.33782	0.19504	11.037	22.074	0.28680	16.003
		71.334	2.96015	1.70904	59.667	119.334	2.51311	68.302
0.53322	28.067	27.216	0.51428	0.29692	16.537	33.074	0.43661	23.587
		62.785	1.94454	1.12268	48.308	96.616	1.65088	58.795
0.62385	31.958	35.286	0.70768	0.40858	22.224	44.448	0.60081	30.998
		54.715	1.41314	0.81588	39.210	78.420	1.19973	50.188
0.66278	33.536	43.072	0.93487	0.53975	28.358	56.716	0.79369	38.439
		46.929	1.06971	0.61760	31.700	63.400	0.90816	42.244
0.64982	33.017	50.801	1.22616	0.70792	35.295	70.590	1.04099	46.151
		39.199	0.81555	0.47086	25.214	50.428	0.69239	34.698
0.58491	30.324	58.701	1.64478	0.94961	43.519	87.038	1.39639	54.392
		31.299	0.60804	0.35105	19.344	38.688	0.51621	27.303
0.46906	25.130	66.990	2.35471	1.35949	53.663	107.326	1.99910	63.425
		23.011	0.42470	0.24520	13.777	27.554	0.36056	19.827
0.30605	17.017	75.826	3.95952	2.28603	66.374	132.748	3.36155	73.433
		14.175	0.25258	0.14583	8.297	16.594	0.21444	12.103
0.10674	6.093	85.198	11.9037	6.87260	81.721	163.442	10.1060	84.349
		4.803	0.08402	0.04851	2.777	5.554	0.07133	4.080

Schedule for the calculation of v_d, v_h and v_v for a

a) Calculations for the tooth spaces

$$\cos r_c = \cos B_{lh} \cos r_1 + \sin B_{lh} \sin r_1 \cos \Phi$$

$$\sin \Phi_c = \frac{\sin r_1}{\sin r_c} \sin \Phi$$

$$\sin v_{vc} = \sin r_c \cos \Phi_c$$

$$\tan v_{hc} = \tan r_c \sin \Phi_c$$

$$\sin (\Theta - \Theta_c) = \frac{\sin B_{lh}}{\sin r_1} \sin \Phi_c$$

$$\tan (v_{dc} - \Theta_c) = \frac{\tan \Phi_c}{\cos r_c}$$

$$v_{dc} = (v_{dc} - \Theta_c) - (\Theta - \Theta_c) + \Theta$$

Tooth space No.	$\cos r_c = \cos B_{lh} \cos r_1 + \sin B_{lh} \sin r_1 \cdot \cos \Phi$	r_c degrees	$\sin r_c$	$\tan r_c$	$= \frac{\sin \Phi_c}{\sin r_1} \cdot \frac{\sin r_c}{\sin \Phi}$	Φ_c degrees	$\cos \Phi_c$	$\tan \Phi_c$	$\sin v_{vc} = \sin r_c \cdot \cos \Phi_c$
1	0.73088	43.040	0.68251	0.93382	0.00000	0.000	1.00000	0.00000	0.68251
2	0.74006	42.263	0.67254	0.90875	0.21491	12.410	0.97663	0.22005	0.65682
3	0.76507	40.087	0.64395	0.84169	0.38325	22.535	0.92364	0.41493	0.59478
4	0.80009	36.861	0.59987	0.74976	0.49443	29.632	0.86922	0.56882	0.52142
5	0.83904	32.961	0.54407	0.64844	0.55916	33.998	0.82906	0.67446	0.45107
6	0.87723	28.690	0.48007	0.54726	0.58870	36.065	0.80835	0.72828	0.38806
7	0.91135	24.307	0.41162	0.45165	0.58842	36.045	0.80855	0.72775	0.33282
8	0.93935	20.057	0.34295	0.36510	0 55628	33.799	0.83099	0.66942	0.28499
9	0.96016	16.228	0.27946	0.29105	0.48170	28.797	0.87633	0.54968	0.24490
10	0.97358	13.200	0.22835	0.23455	0.34459	20.157	0.93875	0.36708	0.21436
11	0.98003	11.470	0.19885	0.20291	0.12896	7.409	0.99165	0.13003	0.19719

b) Calculations for the tips of the teeth

$$\cos r_t = \cos A \cos r_1 - \sin A \sin r_1 \cos \Phi$$

$$\sin \Phi_t = \frac{\sin r_1}{\sin r_t} \sin \Phi$$

$$\sin v_{vt} = \sin r_t \cos \Phi_t$$

$$\tan v_{ht} = \tan r_t \sin \Phi_t$$

$$\sin (\Theta_t - \Theta) = \frac{\sin A}{\sin r_1} \sin \Phi_t$$

$$\tan (v_{dt} - \Theta_t) = \frac{\tan \Phi_t}{\cos r_t}$$

$$v_{dt} = (v_{dt} - \Theta_t) + (\Theta_t - \Theta) + \Theta$$

Tooth No.	$\cos r_t = \cos A \cdot \cos r_1 - \sin A \cdot \sin r_1 \cos \Phi$	r_t degrees	$\sin r_t$	$\tan r_t$	$= \frac{\sin \Phi_t}{\sin r_1} \cdot \frac{\sin r_t}{\sin \Phi}$	Φ_t degrees	$\cos \Phi_t$	$\tan \Phi_t$	$\sin v_{vt} = \sin r_t \cdot \cos \Phi_t$
1½	0.67079	47.872	0.74164	1.10564	0.10183	5.845	0.99480	0.10237	0.73778
2½	0.69258	46.165	0.72134	1.04152	0.28053	16.292	0.95984	0.29227	0.69237
3½	0.72947	43.158	0.68401	0.93768	0.40616	23.964	0.91380	0.44447	0.62505
4½	0.77341	39.339	0.63391	0.81963	0.48087	28.742	0.87679	0.54844	0.55581
5½	0.81823	35.092	0.57489	0.70260	0.51552	31.032	0.85688	0.60162	0.49261
6½	0.85978	30.708	0.51066	0.59394	0.51741	31.159	0.85574	0.60465	0.43699
7½	0.89545	26.434	0.44516	0.49714	0.48848	29.241	0.87257	0.55982	0.38843
8½	0.92373	22.523	0.38306	0.41468	0.42537	25.174	0.90502	0.47001	0.34668
9½	0.94390	19.283	0.33023	0.34986	0.32214	18.792	0.94670	0.34027	0.31263
10½	0.95588	17.083	0.29376	0.30732	0.17626	10.152	0.98434	0.17907	0.28916
11½	0.95982	16.298	0.28064	0.29238	0.00000	0.000	1.00000	0.00000	0.28064

non-circular bevel gear. A. The single-period wheel

$m = 3$ mm

$R_{sph} = 68$ mm

$\sin B_{lk} = \dfrac{m}{R_{sph}} = 0.044118$ or $B_{lk} = 2.529°$

$\cos B_{lk} = 0.99903$

$\Theta = \vartheta_1$

v_{vc} degrees	$\tan v_{kc} = \tan \nu_c \cdot \sin \Phi_c$	v_{kc} degrees	$\sin(\Theta - \Theta_c) = \dfrac{\sin B_{lk}}{\sin \nu_1} \cdot \sin \Phi_c$	$\Theta - \Theta_c$ degrees	$\tan(v_{dc} - \Theta_c) = \dfrac{\tan \Phi_c}{\cos \nu_c}$	$v_{dc} - \Theta_c$ degrees	$v_{dc} = (v_{dc} - \Theta_c) - (\Theta - \Theta_c) + \Theta$ degrees
43.040	0.00000	0.000	0.00000	0.000	0.00000	0.000	0.000
41.058	0.19530	11.051	0.01347	0.772	0.29734	16.559	26.875
36.497	0.32258	17.879	0.02506	1.436	0.54234	28.473	49.111
31.427	0.37070	20.340	0.03460	1.983	0.71095	35.411	66.502
26.813	0.36258	19.930	0.04292	2.460	0.80385	38.794	80.782
22.834	0.32217	17.857	0.05077	2.910	0.83020	39.699	93.505
19.440	0.26576	14.883	0.05840	3.348	0.79854	38.609	105.851
16.558	0.20310	11.481	0.06495	3.724	0.71264	35.475	118.789
14.176	0.14020	7.981	0.06704	3.844	0.57249	29.791	133.273
12.378	0.08082	4.621	0.05654	3.241	0.37704	20.659	150.166
11.373	0.02617	1.499	0.02355	1.349	0.13268	7.558	169.651

$A = \dfrac{180°}{\pi} \cdot \dfrac{m}{R_{sph}} = 2.528°$

$\sin A = 0.044103$

$\cos A = 0.99903$

$\Theta = \vartheta_1$

v_{vt} degrees	$\tan v_{kt} = \tan \nu_t \cdot \sin \Phi_t$	v_{kt} degrees	$\sin(\Theta_t - \Theta) = \dfrac{\sin A}{\sin \nu_1} \cdot \sin \Phi_t$	$\Theta_t - \Theta$ degrees	$\tan(v_{dt} - \Theta_t) = \dfrac{\tan \Phi_t}{\cos \nu_t}$	$v_{dt} - \Theta_t$ degrees	$v_{dt} = (v_{dt} - \Theta_t) + (\Theta_t - \Theta) + \Theta$ degrees
47.543	0.11259	6.424	0.00631	0.362	0.15261	8.677	14.593
43.818	0.29218	16.287	0.01789	1.025	0.42200	22.880	40.499
38.686	0.38085	20.851	0.02738	1.569	0.60931	31.355	60.478
33.766	0.39414	21.512	0.03513	2.013	0.70912	35.341	76.042
29.512	0.36220	19.911	0.04180	2.396	0.73527	36.326	89.150
25.912	0.30731	17.083	0.04771	2.735	0.70326	35.117	101.252
22.857	0.24284	13.649	0.05244	3.006	0.62518	32.013	113.439
20.284	0.17639	10.003	0.05417	3.105	0.50882	26.968	126.689
18.218	0.11270	6.430	0.04885	2.800	0.36049	19.824	141.958
16.808	0.05417	3.101	0.03083	1.767	0.18734	10.611	159.906
16.298	0.00000	0.000	0.00000	0.000	0.00000	0.000	180.000

Schedule for the calculation of v_d, v_h and v_v for a

a) Calculations for the tooth spaces

$$\cos r_c = \cos B_{th} \cos r_2 + \sin B_{th} \sin r_2 \cos \Phi$$

$$\sin \Phi_c = \frac{\sin r_2}{\sin r_c} \sin \Phi$$

$$\sin v_{vc} = \sin r_c \cos \Phi_c$$

$$\tan v_{hc} = \tan r_c \sin \Phi_c$$

$$\sin (\Theta - \Theta_c) = \frac{\sin B_{th}}{\sin r_2} \sin \Phi_c$$

$$\tan (v_{dc} - \Theta_c) = \frac{\tan \Phi_c}{\cos r_c}$$

$$v_{dc} = (v_{dc} - \Theta_c) - (\Theta - \Theta_c) + \Theta$$

Tooth space No.	$\cos r_c = \cos B_{th} \cos r_2 + \sin B_{th} \sin r_2 \cdot \cos \Phi$	r_c degrees	$\sin r_c$	$\tan r_c$	$\sin \Phi_c = \frac{\sin r_2}{\sin r_c} \cdot \sin \Phi$	Φ_c degrees	$\cos \Phi_c$	$\tan \Phi_c$	$\sin v_{vc} = \sin r_c \cdot \cos \Phi_c$
1	0.28066	73.701	0.95981	3.41995	0.00000	0.000	1.00000	0.00000	0.95981
2	0.29368	72.922	0.95591	3.25500	0.20788	11.998	0.97816	0.21252	0.93503
3	0.32984	70.741	0.94404	2.86212	0.37070	21.759	0.92875	0.39914	0.87678
4	0.38249	67.512	0.92396	2.41565	0.47772	28.537	0.87851	0.54379	0.81171
5	0.44456	63.605	0.89575	2.01493	0.53873	32.597	0.84248	0.63946	0.75465
6	0.51009	59.330	0.86012	1.68620	0.56408	34.339	0.82571	0.68315	0.71021
7	0.57441	54.941	0.81856	1.42502	0.55860	33.959	0.82944	0.67347	0.67895
8	0.63352	50.690	0.77373	1.22133	0.52019	31.345	0.85405	0.60909	0.66080
9	0.68377	46.861	0.72970	1.06717	0.44016	26.114	0.89792	0.49020	0.65521
10	0.72124	43.843	0.69269	0.96041	0.30523	17.772	0.95228	0.32053	0.65963
11	0.74166	42.127	0.67078	0.90442	0.11119	6.384	0.99380	0.11188	0.66662

b) Calculations for the tips of the teeth

$$\cos r_t = \cos A \cos r_2 - \sin A \sin r_2 \cos \Phi$$

$$\sin \Phi_t = \frac{\sin r_2}{\sin r_t} \sin \Phi$$

$$\sin v_{vt} = \sin r_t \cos \Phi_t$$

$$\tan v_{ht} = \tan r_t \sin \Phi_t$$

$$\sin (\Theta_t - \Theta) = \frac{\sin A}{\sin r_2} \sin \Phi_t$$

$$\tan (v_{dt} - \Theta_t) = \frac{\tan \Phi_t}{\cos r_t}$$

$$v_{dt} = (v_{dt} - \Theta_t) + (\Theta_t - \Theta) + \Theta$$

Tooth No.	$\cos r_t = \cos A \cdot \cos r_2 - \sin A \cdot \sin r_2 \cdot \cos \Phi$	r_t degrees	$\sin r_t$	$\tan r_t$	$\sin \Phi_t = \frac{\sin r_2}{\sin r_t} \cdot \sin \Phi$	Φ_t degrees	$\cos \Phi_t$	$\tan \Phi_t$	$\sin v_{vt} = \sin r_t \cdot \cos \Phi_t$
1½	0.19882	78.532	0.98004	4.92924	0.10509	6.032	0.99447	0.10567	0.97462
2½	0.22799	76.821	0.97366	4.27057	0.28951	16.829	0.95717	0.30247	0.93196
3½	0.27881	73.811	0.96035	3.44449	0.41940	24.797	0.90780	0.46200	0.87181
4½	0.34225	69.986	0.93961	2.74539	0.49750	29.835	0.86746	0.57352	0.81507
5½	0.41094	65.736	0.91166	2.21847	0.53540	32.371	0.84460	0.63391	0.76999
6½	0.47948	61.349	0.87755	1.83025	0.54094	32.748	0.84106	0.64317	0.73807
7½	0.54358	57.072	0.83936	1.54411	0.51603	31.066	0.85658	0.60243	0.71898
8½	0.59955	53.162	0.80033	1.33488	0.45638	27.154	0.88978	0.51292	0.71212
9½	0.64377	49.926	0.76522	1.18863	0.35280	20.659	0.93570	0 37705	0.71602
10½	0.67250	47.741	0.74011	1.10057	0.19710	11.368	0.98038	0.20105	0.72559
11½	0.68253	46.958	0.73086	1.07079	0.00000	0.000	1.00000	0.00000	0.73086

non-circular bevel gear. B. The two-period wheel

$m = 3$ mm

$R_{sph} = 68$ mm

$\sin B_{th} = \dfrac{m}{R_{sph}} = 0.044118$ or $B_{th} = 2.529°$

$\cos B_{th} = 0.99903$

$\Theta = \vartheta_2$

v_{vc} degrees	$\tan v_{hc} = \tan r_c \cdot \sin \Phi_c$	v_{hc} degrees	$\sin(\Theta - \Theta_c) = \dfrac{\sin B_{th}}{\sin r_2} \cdot \sin \Phi_c$	$\Theta - \Theta_c$ degrees	$\tan(v_{dc} - \Theta_c) = \dfrac{\tan \Phi_c}{\cos r_c}$	$v_{dc} - \Theta_c$ degrees	$v_{dc} = (v_{dc} - \Theta_c) - (\Theta - \Theta_c) + \Theta$ degrees
73.700	0.00000	0.000	0.00000	0.000	0.00000	0.000	0.000
69.233	0.67665	34.084	0.00948	0.543	0.72364	35.891	43.471
61.256	1.06099	46.695	0.01709	0.979	1.21010	50.430	65.454
54.263	1.15400	49.090	0.02247	1.287	1.42171	54.878	77.178
48.995	1.08550	47.348	0.02607	1.493	1.43841	55.192	84.697
45.252	0.95115	43.566	0.02834	1.623	1.33927	53.252	90.068
42.762	0.79602	38.520	0.02937	1.682	1.17246	49.539	94.008
41.361	0.63532	32.429	0.02879	1.649	0.96144	43.874	96.617
40.935	0.46973	25.161	0.02568	1.471	0.71691	35.637	97.591
41.272	0.29315	16.338	0.01864	1.068	0.44442	23.961	96.326
41.807	0.10056	5.742	0.00698	0.400	0.15085	8.578	92.527

$A = \dfrac{180°}{\pi} \cdot \dfrac{m}{R_{sph}} = 2.528°$

$\sin A = 0.044103$

$\cos A = 0.99903$

$\Theta = \vartheta_2$

v_{vt} degrees	$\tan v_{ht} = \tan r_t \cdot \sin \Phi_t$	v_{ht} degrees	$\sin(\Theta_t - \Theta) = \dfrac{\sin A}{\sin r_2} \cdot \sin \Phi_t$	$\Theta_t - \Theta$ degrees	$\tan(v_{dt} - \Theta_t) = \dfrac{\tan \Phi_t}{\cos r_t}$	$v_{dt} - \Theta_t$ degrees	$v_{dt} = (v_{dt} - \Theta_t) + (\Theta_t - \Theta) + \Theta$ degrees
77.063	0.51801	27.385	0.00478	0.274	0.53149	27.990	32.344
68.743	1.23647	51.036	0.01326	0.760	1.32668	52.993	65.856
60.670	1.44462	55.308	0.01950	1.117	1.65710	58.891	79.835
54.594	1.36583	53.790	0.02370	1.358	1.67573	59.173	87.834
50.353	1.18777	49.905	0.02636	1.511	1.54259	57.046	93.255
47.568	0.99006	44.714	0.02777	1.591	1.34139	53.296	97.131
45.970	0.79681	38.548	0.02781	1.594	1.10826	47.939	99.721
45.408	0.60921	31.350	0.02593	1.486	0.85551	40.547	100.828
45.727	0.41935	22.751	0.02108	1.208	0.58569	30.357	99.867
46.518	0.21692	12.239	0.01224	0.701	0.29896	16.645	96.144
46.958	0.00000	0.000	0.00000	0.000	0.00000	0.000	90.000

Schedule for the calculation of various data for a non-circular bevel gear

A) Calculation of requisite milling cutters

$$(Z_c)_n \leq \frac{360^\circ}{\Delta\Phi + \Delta\vartheta \cos (r)_n} \text{ where } \Delta\Phi = \Phi_{n+\frac{1}{2}} - \Phi_{n-\frac{1}{2}} \text{ and } \Delta\vartheta = \vartheta_{n+\frac{1}{2}} - \vartheta_{n-\frac{1}{2}}$$

Tooth space No.	The single-period wheel						The two-period wheel				
	[1] $\Delta\Phi$ degrees	[1] $\Delta\vartheta$ degrees	[1] $\cos r_1$	$\Delta\Phi + \Delta\vartheta \cos r_1$ degrees	$Z_c \leq$	Choose a cutter for $Z_c =$	[1] $\Delta\vartheta$ degrees	[1] $\cos r_2$	$\Delta\Phi + \Delta\vartheta \cos r_2$ degrees	$Z_c \leq$	Choose a cutter for $Z_c =$
1	12.186	11.108	0.7001	19.96	18.1	17	8.160	0.2380	14.13	25.5	21
2	10.924	11.040	0.7104	18.77	19.2		8.023	0.2521	12.95	27.8	26
3	8.113	10.960	0.7381	16.20	22.2	21	7.724	0.2908	10.36	34.8	
4	5.194	11.134	0.7763	13.84	26.0		7.476	0.3463	7.78	46.3	35
5	2.693	11.740	0.8183	12.30	29.3	26	7.395	0.4108	5.73	62.8	55
6	0.519	12.972	0.8593	11.67	30.9		7.546	0.4783	4.13	87.2	
7	— 1.578	15.020	0.8958	11.88	30.3		7.944	0.5439	2.74	131.3	
8	— 3.891	18.196	0.9259	12.96	27.8		8.607	0.6038	1.31	275	135
9	— 6.612	22.718	0.9484	14.93	24.1	21	9.507	0.6542	— 0.39	— 923	∞
10	— 9.605	28.194	0.9632	17.55	20.5	17	10.496	0.6914	— 2.39	— 151	
11	— 11.850	32.472	0.9704	19.66	18.3		11.202	0.7115	— 3.88	— 92.8	∞ [2]

[1]) $\Delta\Phi$ and $\Delta\vartheta$ can be computed from Appendix 13 ($\Delta\Phi_{-\frac{1}{2}} = -\Delta\Phi_{1\frac{1}{2}}$), cos r is taken from Appendix 12.

[2]) Strictly, a cutter designed for $Z_c = -135$ (i.e. for 135 internal teeth) should be chosen. As, however, no such cutter was available, a cutter for $Z_c = \infty$ was used, which is also possible.

B) Calculations of the out-of-centre setting b_{so} and and the additional rotation of the rotary milling table Δv_{hc} when milling the side cuts

For the calculation of b_{so} use App. 8

$d = 6.47$ mm for all cutters used

$$\frac{\text{Length of tooth face}}{\text{Apex distance}} = \frac{F}{R_{sph}} = \frac{1}{6}$$

$$\Delta v_{hc} = \frac{m}{R_{sph} \cos B_{th}} \left[45^\circ - \frac{180^\circ}{\pi} c_c - \frac{\pi^2}{8} \frac{15^\circ}{Z_c^2} \right]$$

$R_{sph} = 68$ mm

$\cos B_{th} = 0.99903$, $m = 3$ mm

For Z_c the theoretical value should be used, but if, as below, the value for the cutter used is taken, the error will be less than 0.001°.

Cutter for number of teeth	c_{pc}	Chord thickness g_p at pitch line: Height setting of calliper h_{pc} mm	Chord thickness g_p at pitch line: Measured value of g_p mm	c_c	b_{so} mm	$c_c \frac{180^\circ}{\pi} = 57.2958^\circ c_c$ degrees	$\frac{15^\circ \cdot \pi^2}{8 Z_c^2} = \frac{18.5055^\circ}{Z_c^2}$ degrees	$45^\circ - 57.2958^\circ c_c - \frac{18.5055^\circ}{Z_c^2}$ degrees	$\Delta v_{hc} = 0.04416 \cdot$ the value in the preceding column degrees
17	1.036	3.362	2.90	0.419	0.193	24.007	0.064	20.929	0.924
21	1.029	3.383	2.90	0.409	0.223	23.434	0.042	21.524	0.950
26	1.024	3.398	2.90	0.400	0.250	22.918	0.027	22.055	0.974
35	1.018	3.416	2.96	0.392	0.304	22.460	0.015	22.525	0.995
55	1.011	3.437	2.96	0.381	0.337	21.830	0.006	23.164	1.023
135	1.005	3.455	2.96	0.372	0.364	21.314	0.001	23.685	1.046
∞	1.000	3.470	2.96	0.364	0.388	20.856	0	24.144	1.066

C) Calculation of f, D_c and D_t

$D_c = 2\,[2\,m + f \sin (v_{vc})_{max}]$

$D_t = 2\,f \sin (v_{vt})_{max}$

f = distance measured from the centre of the dividing head to the locating face on the arbor + distance from the locating face on the wheel boss to the apex of the pitch cone taken from the drawing (compare Fig. 18).

a) The single-period wheel

$f = 118.12$ mm + 70.00 mm = 188.12 mm

$D_c = 2\,[6 + 188.12 \cdot 0.68251]$ mm = 268.79 mm

$D_t = 2 \cdot 188.12 \cdot 0.73778$ mm = 277.59 mm

b) The two-period wheel

$f = 107.62$ mm + 58.00 mm = 165.62 mm

$D_c = 2\,[6 + 165.62 \cdot 0.95981]$ mm = 329.93 mm

$D_t = 2 \cdot 165.62 \cdot 0.97462$ mm = 322.84 mm

Appendix 17

Schedule for the calculation of l_t and h_t for cutting the tooth tips of the wheels in a non-circular bevel gear

$$l_t = f \cos v_{vt} \sin v_{ht},\ h_t = \frac{D_t}{2} - f \sin v_{vt}$$

A) The single-period wheel

From Appendix 16: $f = 188.12$ mm, $\frac{D_t}{2} = 138.79$ mm

Tooth No.	cos v_{vt} [1]	sin v_{ht} [1]	$f \cos v_{vt}$ mm	$l_t = f \cos v_{vt} \cdot \sin v_{ht}$ mm	$f \sin v_{vt}$ [1] mm	$h_t = \frac{D_t}{2} - f \sin v_{vt}$ mm
1½	0.67504	0.11189	126.989	14.209	138.791	0.00
2½	0.72154	0.28045	135.736	38.067	130.249	8.54
3½	0.78058	0.35594	146.843	52.267	117.584	21.21
4½	0.83132	0.36669	156.388	57.346	104.559	34.23
5½	0.87025	0.34056	163.711	55.753	92.670	46.12
6½	0.89947	0.29376	169.208	49.707	82.207	56.58
7½	0.92148	0.23597	173.349	40.905	73.071	65.72
8½	0.93799	0.17370	176.455	30.650	65.217	73.57
9½	0.94987	0.11199	178.690	20.011	58.812	79.98
10½	0.95728	0.05410	180.084	9.743	54.397	84.39
11½	0.95982	0.00000	180.561	0.000	52.794	86.00

[1]) v_{vt}, v_{ht} and sin v_{vt} from Appendix 14

B) The two-period wheel

From Appendix 16: $f = 165.62$ mm, $\frac{D_t}{2} = 161.42$ mm

Tooth No.	cos v_{vt} [2]	sin v_{ht} [2]	$f \cos v_{vt}$ mm	$l_t = f \cos v_{vt} \cdot \sin v_{ht}$ mm	$f \sin v_{vt}$ [2] mm	$h_t = \frac{D_t}{2} - f \sin v_{vt}$ mm
1½	0.22388	0.45997	37.079	17.055	161.417	0.00
2½	0.36255	0.77755	60.046	46.689	154.351	7.07
3½	0.48984	0.82222	81.127	66.704	144.389	17.03
4½	0.57936	0.80686	95.954	77.421	134.992	26.43
5½	0.63806	0.76498	105.675	80.839	127.526	33.89
6½	0.67472	0.70357	111.747	78.622	122.239	39.18
7½	0.69503	0.62317	115.111	71.734	119.077	42.34
8½	0.70205	0.52026	116.274	60.493	117.941	43.48
9½	0.69808	0.38673	115.616	44.712	118.587	42.83
10½	0.68813	0.21199	113.968	24.160	120.172	41.25
11½	0.68254	0.00000	113.042	0.000	121.045	40.38

[2]) v_{vt}, v_{ht} and sin v_{vt} from Appendix 15

Schedule for the calculation of v_h, l and h_c for cutting

$l_{cL} = f \sin (v_{hc} - \Delta v_{hc}) \cos v_{vc} + b_{so}$, $l_c = f \cos v_{vc} \sin v_{hc}$, $l_{cR} = f \sin (v_{hc} + \Delta v_{hc}) \cos v_{vc} - b_{so}$,

A) The single-period wheel

$f = 188.12$ mm, $D_c = 268.79$ mm from Appendix 16, $\frac{D_c}{2} - 2\,m = 128.40$ mm

Tooth space No.	Number of teeth for cutter used [1]	Δv_{hc} [1]	b_{so} [1]	v_h			sin v_{hcL}	sin v_{hc}
				Left-hand cut $v_{hcL} = v_{hc} - \Delta v_{hc}$ [3]	Central cut v_{hc} [2]	Right-hand cut $v_{hcR} = v_{hc} + \Delta v_{hc}$ [3]		
		degrees	mm	degrees	degrees	degrees		
1	17	0.924	0.193	— 0.924	0.000	0.924	— 0.01613	0.00000
2				10.127	11.051	11.975	0.17583	0.19169
3	21	0.950	0.223	16.929	17.879	18.829	0.29118	0.30700
4				19.390	20.340	21.290	0.33200	0.34759
5	26	0.974	0.250	18.956	19.930	20.904	0.32484	0.34087
6				16.883	17.857	18.831	0.29042	0.30664
7				13.909	14.883	15.857	0.24038	0.25685
8				10.507	11.481	12.455	0.18236	0.19905
9	21	0.950	0.223	7.031	7.981	8.931	0.12241	0.13885
10	17	0.924	0.193	3.697	4.621	5.545	0.06448	0.08057
11				0.575	1.499	2.423	0.01004	0.02616

[1]) Taken from Appendix 16. [2]) Taken from Appendix 14. [3]) Δv_{hc} from Appendix 16.

B) The two-period wheel

$f = 165.62$ mm, $D_c = 329.93$ mm from Appendix 16, $\frac{D_c}{2} - 2\,m = 158.97$ mm

Tooth space No.	Number of teeth for cutter used [5]	Δv_{hc} [5]	b_{so} [5]	v_h			sin v_{hcL}	sin v_{hc}
				Left-hand cut $v_{hcL} = v_{hc} - \Delta v_{hc}$ [7]	Central cut v_{hc} [6]	Right-hand cut $v_{hcR} = v_{hc} + \Delta v_{hc}$ [7]		
		degrees	mm	degrees	degrees	degrees		
1	21	0.950	0.223	— 0.950	0.000	0.950	— 0.01658	0.00000
2	26	0.974	0.250	33.110	34.084	35.058	0.54625	0.56041
3				45.721	46.695	47.669	0.71595	0.72771
4	35	0.995	0.304	48.095	49.090	50.085	0.74425	0.75574
5	55	1.023	0.337	46.325	47.348	48.371	0.72327	0.73549
6				42.543	43.566	44.589	0.67614	0.68919
7				37.497	38.520	39.543	0.60872	0.62279
8	135	1.046	0.364	31.383	32.429	33.475	0.52076	0.53626
9	∞	1.066	0.388	24.095	25.161	26.227	0.40825	0.42517
10				15.272	16.338	17.404	0.26340	0.28131
11				4.676	5.742	6.808	0.08152	0.10005

[5]) Taken from Appendix 16. [6]) Taken from Appendix 15. [7]) Δv_{hc} from Appendix 16.

the tooth spaces of a non-circular bevel gear

$v_{hcL} = v_{hc} - \Delta v_{hc}$, $v_{hcR} = v_{hc} + \Delta v_{hc}$, $h_c = \frac{D_e}{2} - 2\,m - f \sin v_{vc}$

$\sin v_{hcR}$	$\cos v_{vc}$	$f \cos v_{vc}$ mm	l			[4]) $f \sin v_{vc}$ mm	h_c
			Left-hand cut $l_{cL} = f \cos v_{vc} \cdot \sin v_{hcL} + b_{so}$ mm	Central cut $l_c = f \cos v_{vc} \cdot \sin v_{hc}$ mm	Right-hand cut $l_{cR} = f \cos v_{vc} \cdot \sin v_{hcR} - b_{so}$ mm		$h_c = \frac{D_e}{2} - 2\,m - f \sin v_{vc}$ mm
0.01613	0.73088	137.493	— 2.025	0.000	2.025	128.39	0.00
0.20749	0.75404	141.850	25.134	27.191	29.239	123.56	4.84
0.32274	0.80389	151.228	44.258	46.427	48.584	111.89	16.51
0.36309	0.85331	160.525	53.517	55.797	58.062	98.09	30.31
0.35680	0.89249	167.895	54.789	57.230	59.655	84.86	43.54
0.32278	0.92163	173.377	50.602	53.164	55.713	73.00	55.40
0.27324	0.94299	177.395	42.892	45.564	48.221	62.61	65.79
0.21568	0.95853	180.319	33.133	35.892	38.641	53.61	74.79
0.15525	0.96955	182.392	22.550	25.325	28.093	46.07	82.33
0.09663	0.97676	183.748	12.041	14.805	17.563	40.33	88.07
0.04227	0.98036	184.425	2.045	4.825	7.603	37.10	91.30

[4]) $\sin v_{vc}$ from Appendix 14.

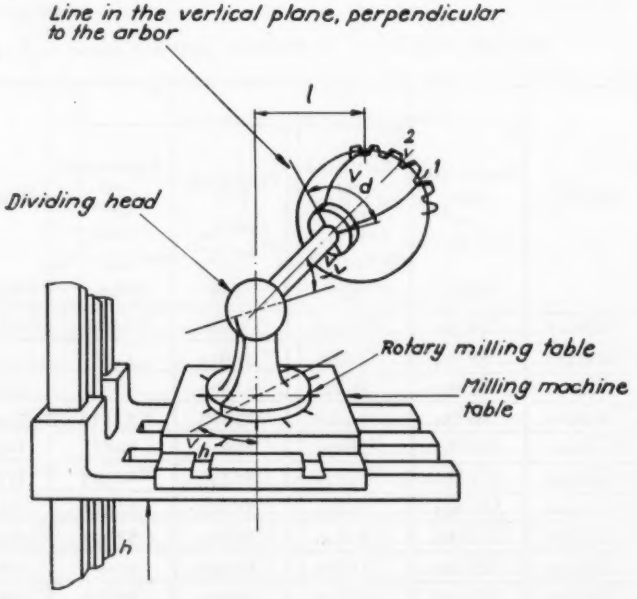
$\sin v_{hcR}$	$\cos v_{vc}$	$f \cos v_{vc}$ mm	l			[8]) $f \sin v_{vc}$ mm	h_c
			Left-hand cut $l_{cL} = f \cos v_{vc} \cdot \sin v_{hcL} + b_{so}$ mm	Central cut $l_c = f \cos v_{vc} \cdot \sin v_{hc}$ mm	Right-hand cut $l_{cR} = f \cos v_{vc} \cdot \sin v_{hcR} - b_{so}$ mm		$h_c = \frac{D_e}{2} - 2\,m - f \sin v_{vc}$ mm
0.01658	0.28067	46.485	— 0.548	0.000	0.548	158.96	0.00
0.57440	0.35457	58.724	32.328	32.910	33.481	154.86	4.11
0.73927	0.48090	79.647	57.273	57.960	58.631	145.21	13.76
0.76700	0.58407	96.734	72.298	73.106	73.891	134.44	24.53
0.74746	0.65612	108.667	78.933	79.923	80.887	124.99	33.98
0.70202	0.70399	116.595	79.172	80.356	81.515	117.62	41.35
0.63666	0.73418	121.595	74.354	75.728	77.078	112.45	46.52
0.55158	0.75056	124.308	65.099	66.661	68.202	109.44	49.53
0.44193	0.75545	125.118	51.467	53.196	54.905	108.52	50.45
0.29911	0.75159	124.478	33.176	35.017	36.845	109.25	49.72
0.11854	0.74540	123.453	10.452	12.351	14.246	110.41	48.56

[8]) $\sin v_{vc}$ from Appendix 15.

Appendix 19

Cutting schedule for the tooth spaces of a single-period non-circular bevel gear wheel

Tooth space No.	v_d in revolutions + degrees on the index plate		v_h in revolutions + degrees on the index plate						v_v	h	l			Choose cutter for number of teeth
			Left-hand cut		Central cut		Right-hand cut				Left-hand cut	Central cut	Right-hand cut	
	rev.	degr.	rev.	degr.	rev.	degr.	rev.	degr.	degr.	mm	mm	mm	mm	
1	0	0.0	59	304.6	60	0.0	60	55.4	43.04	0.00	— 2.03	0.00	2.03	17
2	2	355.0	61	247.6	61	303.1	61	358.5	41.06	4.84	25.13	27.19	29.24	
3	5	164.4	62	295.7	62	352.7	63	49.7	36.50	16.51	44.26	46.43	48.58	21
4	7	140.1	63	83.4	63	140.4	63	197.4	31.43	30.31	53.52	55.80	58.06	
5	8	351.3	63	57.4	63	115.8	63	174.2	26.81	43.54	54.79	57.23	59.66	
6	10	140.2	62	293.0	62	351.4	63	49.9	22.83	55.40	50.60	53.16	55.71	26
7	11	274.0	62	114.5	62	173.0	62	231.4	19.44	65.79	42.89	45.56	48.22	
8	13	71.6	61	270.4	61	328.9	62	27.3	16.56	74.79	33.13	35.89	38.64	
9	14	290.9	61	61.9	61	118.9	61	175.9	14.18	82.33	22.55	25.33	28.09	21
10	16	246.6	60	221.8	60	277.3	60	332.7	12.38	88.07	12.04	14.81	17.56	
11	18	306.0	60	34.5	60	89.9	60	145.4	11.37	91.30	2.05	4.83	7.60	17
12	21	54.0	59	214.6	59	270.1	59	325.5	11.37	91.30	— 7.60	— 4.83	— 2.05	
13	23	113.4	59	27.3	59	82.7	59	138.2	12.38	88.07	— 17.56	— 14.81	— 12.04	
14	25	69.1	58	184.1	58	241.1	58	298.1	14.18	82.33	— 28.09	— 25.33	— 22.55	21
15	26	288.4	57	332.7	58	31.1	58	89.6	16.56	74.79	— 38.64	— 35.89	— 33.13	
16	28	86.0	57	128.6	57	187.0	57	245.5	19.44	65.79	— 48.22	— 45.56	— 42.89	26
17	29	219.8	56	310.1	57	8.6	57	67.0	22.83	55.40	— 55.71	— 53.16	— 50.60	
18	31	8.7	56	185.8	56	244.2	56	302.6	26.81	43.54	— 59.66	— 57.23	— 54.79	
19	32	219.9	56	162.6	56	219.6	56	276.6	31.43	30.31	— 58.06	— 55.80	— 53.52	21
20	34	195.6	56	310.3	57	7.3	57	64.3	36.50	16.51	— 48.58	— 46.43	— 44.26	
21	37	5.0	58	1.5	58	56.9	58	112.4	41.06	4.84	— 29.24	— 27.19	— 25.13	17



Appendix 20

Cutting schedule for the tooth spaces of a two-period non-circular bevel gear wheel

Tooth space No.	v_d in revolutions + degrees on the index plate		v_h in revolutions + degrees on the index plate						v_v	h	l			Choose cutter for number of teeth
			Left-hand cut		Central cut		Right-hand cut				Left-hand cut	Central cut	Right-hand cut	
	rev.	degr.	rev.	degr.	rev.	degr.	rev.	degr.	degr.	mm	mm	mm	mm	
1	0	0.0	59	303.0	60	0.0	60	57.0	73.70	0.00	— 0.55	0.00	0.55	21
2	4	298.8	65	186.6	65	245.0	65	303.5	69.23	4.11	32.33	32.91	33.48	26
3	7	98.2	67	223.3	67	281.7	67	340.1	61.26	13.76	57.27	57.96	58.63	
4	8	207.1	68	5.7	68	65.4	68	125.1	54.26	24.53	72.30	73.11	73.89	35
5	9	147.9	67	259.5	67	320.9	68	22.3	49.00	33.98	78.93	79.92	80.89	55
6	10	2.7	67	32.6	67	94.0	67	155.3	45.25	41.35	79.17	80.36	81.52	
7	10	160.3	66	89.8	66	151.2	66	212.6	42.76	46.52	74.35	75.73	77.08	
8	10	264.7	65	83.0	65	145.7	65	208.5	41.36	49.53	65.10	66.66	68.20	135
9	10	303.6	64	5.7	64	69.7	64	133.6	40.94	50.45	51.47	53.20	54.91	∞
10	10	253.0	62	196.3	62	260.3	62	324.2	41.27	49.72	33.18	35.02	36.85	
11	10	101.1	60	280.6	60	344.5	61	48.5	41.81	48.56	10.45	12.35	14.25	
12	9	258.9	58	311.5	59	15.5	59	79.4	41.81	48.56	— 14.25	— 12.35	— 10.45	
13	9	107.0	57	35.8	57	99.7	57	163.7	41.27	49.72	— 36.85	— 35.02	— 33.18	
14	9	56.4	55	226.4	55	290.3	55	354.3	40.94	50.45	— 54.91	— 53.20	— 51.47	
15	9	95.3	54	151.5	54	214.3	54	277.0	41.36	49.53	— 68.20	— 66.66	— 65.10	135
16	9	199.7	53	147.4	53	208.8	53	270.2	42.76	46.52	— 77.08	— 75.73	— 74.35	55
17	9	357.3	52	204.7	52	266.0	52	327.4	45.25	41.35	— 81.52	— 80.36	— 79.17	
18	10	212.1	51	337.7	52	39.1	52	100.5	49.00	33.98	— 80.89	— 79.92	— 78.93	
19	11	152.9	51	234.9	51	294.6	51	354.3	54.26	24.53	— 73.89	— 73.11	— 72.30	35
20	12	261.8	52	19.9	52	78.3	52	136.7	61.26	13.76	— 58.63	— 57.96	— 57.27	26
21	15	61.2	54	56.5	54	115.0	54	173.4	69.23	4.11	— 33.48	— 32.91	— 32.33	
22	20	0.0	59	303.0	60	0.0	60	57.0	73.70	0.00	— 0.55	0.00	0.55	21
23	24	298.8	65	186.6	65	245.0	65	303.5	69.23	4.11	32.33	32.91	33.48	26
24	27	98.2	67	223.3	67	281.7	67	340.1	61.26	13.76	57.27	57.96	58.63	
25	28	207.1	68	5.7	68	65.4	68	125.1	54.26	24.53	72.30	73.11	73.89	35
26	29	147.9	67	259.5	67	320.9	68	22.3	49.00	33.98	78.93	79.92	80.89	55
27	30	2.7	67	32.6	67	94.0	67	155.3	45.25	41.35	79.17	80.36	81.52	
28	30	160.3	66	89.8	66	151.2	66	212.6	42.76	46.52	74.35	75.73	77.08	
29	30	264.7	65	83.0	65	145.7	65	208.5	41.36	49.53	65.10	66.66	68.20	135
30	30	303.6	64	5.7	64	69.7	64	133.6	40.94	50.45	51.47	53.20	54.91	∞
31	30	253.0	62	196.3	62	260.3	62	324.2	41.27	49.72	33.18	35.02	36.85	
32	30	101.1	60	280.6	60	344.5	61	48.5	41.81	48.56	10.45	12.35	14.25	
33	29	258.9	58	311.5	59	15.5	59	79.4	41.81	48.56	— 14.25	— 12.35	— 10.45	
34	29	107.0	57	35.8	57	99.7	57	163.7	41.27	49.72	— 36.85	— 35.02	— 33.18	
35	29	56.4	55	226.4	55	290.3	55	354.3	40.94	50.45	— 54.91	— 53.20	— 51.47	
36	29	95.3	54	151.5	54	214.3	54	277.0	41.36	49.53	— 68.20	— 66.66	— 65.10	135
37	29	199.7	53	147.4	53	208.7	53	270.2	42.76	46.52	— 77.08	— 75.73	— 74.35	55
38	29	357.3	52	204.7	52	266.0	52	327.4	45.25	41.35	— 81.52	— 80.36	— 79.17	
39	30	212.1	51	337.7	52	39.1	52	100.5	49.00	33.98	— 80.89	— 79.92	— 78.93	
40	31	152.9	51	234.9	51	294.6	51	354.3	54.26	24.53	— 73.89	— 73.11	— 72.30	35
41	32	261.8	52	19.9	52	78.3	52	136.7	61.26	13.76	— 58.63	— 57.96	— 57.27	26
42	35	61.2	54	56.5	54	115.0	54	173.4	69.23	4.11	— 33.48	— 32.91	— 32.33	

See the figure in Appendix 19

Schedule for cutting the contour of non-circular bevel gear wheels

A) The single-period wheel

Tooth No.	v_d in revolutions + degrees on the index plate		v_k in revolutions + degrees on the index plate		v_v	h	l
	rev.	degr.	rev.	degr.	degr.	mm	mm
1	1	223.7	61	25.4	47.54	0.00	14.21
2	4	180.0	62	257.2	43.82	8.54	38.07
3	6	259.1	63	171.1	38.69	21.21	52.27
4	8	161.7	63	210.7	33.77	34.23	57.35
5	9	326.0	63	114.7	29.51	46.12	55.75
6	11	90.1	62	305.0	25.91	56.58	49.71
7	12	217.6	62	98.9	22.86	65.72	40.91
8	14	27.6	61	240.2	20.28	73.57	30.65
9	15	278.3	61	25.8	18.22	79.98	20.01
10	17	276.2	60	186.1	16.81	84.39	9.74
11	20	0	60	0.0	16.30	86.00	0.00
12	22	83.8	59	173.9	16.81	84.39	— 9.74
13	24	81.7	58	334.2	18.22	79.98	— 20.01
14	25	332.4	58	119.8	20.28	73.57	— 30.65
15	27	142.4	57	261.1	22.86	65.72	— 40.91
16	28	269.9	57	55.0	25.91	56.58	— 49.71
17	30	34.0	56	245.3	29.51	46.12	— 55.75
18	31	198.3	56	149.3	33.77	34.23	— 57.35
19	33	100.9	56	188.9	38.69	21.21	— 52.27
20	35	180.0	57	102.8	43.82	8.54	— 38.07
21	38	136.3	58	334.6	47.54	0.00	— 14.21

B) The two-period wheel

Tooth No.	v_d in revolutions + degrees on the index plate		v_k in revolutions + degrees on the index plate		v_v	h	l
	rev.	degr.	rev.	degr.	degr.	mm	mm
1	3	213.8	64	203.1	77.06	0.00	17.06
2	7	114.2	68	182.2	68.74	7.07	46.69
3	8	313.4	69	78.5	60.67	17.03	66.70
4	9	273.4	68	347.4	54.59	26.43	77.42
5	10	130.2	68	114.3	50.35	33.89	80.84
6	10	285.2	67	162.8	47.57	39.18	78.62
7	11	28.8	66	152.9	45.97	42.34	71.73
8	12	113.1	65	81.0	45.41	43.48	60.49
9	11	34.7	63	285.1	45.73	42.83	44.71
10	10	245.8	62	14.3	46.52	41.25	24.16
11	10	0.0	60	0.0	46.96	40.38	0.00
12	9	114.2	57	345.7	46.52	41.25	— 24.16
13	8	325.3	56	74.9	45.73	42.83	— 44.71
14	7	246.9	54	279.0	45.41	43.48	— 60.49
15	8	331.2	53	207.1	45.97	42.34	— 71.73
16	9	74.8	52	197.2	47.57	39.18	— 78.62
17	9	229.8	51	245.7	50.35	33.89	— 80.84
18	10	86.6	51	12.6	54.59	26.43	— 77.42
19	11	46.6	50	281.5	60.67	17.03	— 66.70
20	12	245.8	51	177.8	68.74	7.07	— 46.69
21	16	146.2	55	156.9	77.06	0.00	— 17.06
22	23	213.8	64	203.1	77.06	0.00	17.06
23	27	114.2	68	182.2	68.74	7.07	46.69
24	28	313.4	69	78.5	60.67	17.03	66.70
25	29	273.4	68	347.4	54.59	26.43	77.42
26	30	130.2	68	114.3	50.35	33.89	80.84
27	30	285.2	67	162.8	47.57	39.18	78.62
28	31	28.8	66	152.9	45.97	42.34	71.73
29	32	113.1	65	81.0	45.41	43.48	60.49
30	31	34.7	63	285.1	45.73	42.83	44.71
31	30	245.8	62	14.3	46.52	41.25	24.16
32	30	0.0	60	0.0	46.96	40.38	0
33	29	114.2	57	345.7	46.52	41.25	— 24.16
34	28	325.3	56	74.9	45.73	42.83	— 44.71
35	27	246.9	54	279.0	45.41	43.48	— 60.49
36	28	331.2	53	207.1	45.97	42.34	— 71.73
37	29	74.8	52	197.2	47.57	39.18	— 78.62
38	29	229.8	51	245.7	50.35	33.89	— 80.84
39	30	86.6	51	12.6	54.59	26.43	— 77.42
40	31	46.6	50	281.5	60.67	17.03	— 66.70
41	32	245.8	51	177.8	68.74	7.07	— 46.69
42	36	146.2	55	156.9	77.06	0.00	— 17.06

See the figure in Appendix 19

BIBLIOGRAPHY

1. LEGENDRE, A. M.: *Traité des fonctions elliptiques*. Vol. 1—2, Paris 1826.
2. SCHÖNEMANN, T.: *Die geometrischen Constructionen der ebenen und konischen exzentrischen Rad- und Zahn-Curven*. Berlin 1842, 75 pp. + 3 diagrams.
3. DISTELI, M.: *Über Rollkurven und Rollflächen*. Zt. Math. & Phys. 43 (1898), pp. 1—35.
4. REULEAUX, F.: *Lehrbuch der Kinematik*. Vol. 2, Braunschweig 1900, 788 pp. (see p. 472).
5. FLANDERS, R. E.: *Cutting bevel gear teeth — A new method of obtaining the set-over*. Machinery NY 16 (1909) December, pp. 259—62.
6. HEUMAN, C.: *Elliptiska integraler i dynamiska problem*. (Elliptic integrals in dynamic problems). Skrifter, utgivna av Kungl. Tekniska Högskolan (Papers issued by the Royal Swedish Institute of Technology), Stockholm 1918, pp. 201—220.
7. *Practical treatise on gearing*. Brown & Sharpe Manufacturing Company, 16 ed., Providence, R. I., 1929, 243 pp. (see pp. 82—86).
8. PETERS, J.: *Sechsstellige Werte der Kreis- und Evolventenfunktionen von Hundertstel zu Hundertstel des Grades nebst einigen Hilfstafeln für die Zahnradtechnik*. Berlin und Bonn 1937, 217 pp.
9. PESQUEIRA, J. J.: *Principles of design for non-circular gears*. 1—8. Machinist 81 (1937), pp. 145E—46E, 174—75, 219—21, 257—58, 295—96, 336—37, 381—83, 455E—56E (see pp. 257—58).
10. *Barlow's Tables*. Edited by L. J. Comrie. Fourth edition, London 1941, 258 pp.
11. HEUMAN, C.: *Tables of complete elliptic integrals*. Journal of Mathematics and Physics. Vol. 20 (1941): No. 2, pp. 127—206, 336.
12. OLSSON, U.: *Non-circular cylindrical gears*. Stockholm 1953, 216 pp. (Acta Polytechnica No. 135. Mechanical Engineering Series. Vol. 2. No. 10).

For further references see also the bibliography in »Non-circular cylindrical gears» (Ref. 12).

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